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On the Fluid-Dynamical Approximation to the Boltzmann Equation at the Level of the Navier-Stokes Equation

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Abstract. The compressible and heat-conductive Navier-Stokes equation obtained as the second approximation of the formal Chapman-Enskog expansion is investigated on its relations to the original nonlinear Boltzmann equation and also to the incompressible Navier-Stokes equation. The solutions of the Boltzmann equation and the incompressible Navier-Stokes equation for small initial data are proved to be asymptotically equivalent (mod decay rate $t^{-5/4}$) as $t \to +\infty$ to that of the compressible Navier-Stokes equation for the corresponding initial data.

1. Introduction

The nonlinear Boltzmann equation for a rarefied simple gas is given in the form

$$F_t + v^j F_{x_j} = \frac{1}{\varepsilon} Q(F, F) \tag{1.1}$$

where $t \geqq 0$: time, $x \in R^3$: physical space, $v \in R^3$: velocity space, ε: mean free path, $F = F(t, x, v)$ is the mass density distribution function and Q represents the quadratic collision operator. Here and in what follows, we use the summation convention when we are not confused. Let us introduce the fluid-dynamical quantities as follows:

mass density: $\varrho \equiv \int F(t, x, v)\,dv$,

fluid flow velocity: $u^i \equiv \frac{1}{\varrho} \int v^i F(t, x, v)\,dv$,

momentum: $m^i \equiv \varrho u^i$,

pressure tensor: $P^{ij} \equiv \int c^i c^j F(t, x, v)\,dv$,

pressure: $p \equiv \frac{1}{3} P^{kk}$,

viscous term: $p^{ij} \equiv P^{ij} - p\delta^{ij}$,

heat flow vector: $q^i \equiv \frac{1}{2} \int c^i |c|^2 F(t, x, v)\,dv$,