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Notes on Algebraic Invariants for Non-commutative Dynamical Systems

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Abstract. We consider an algebraic invariant for non-commutative dynamical systems naturally arising as the spectrum of the modular operator associated to an invariant state, provided certain conditions of mixing type are present. This invariant turns out to be exactly the annihilator of the invariant T of Connes. Further comments are included, in particular on the type of certain algebras of local observables.

1. Preliminaries

Following a standard terminology we say that a triple $\{\mathscr{R}, G, \alpha\}$ is W^*-system if $\mathscr{R}$ is a von Neumann algebra, G is a locally compact (Hausdorff) group and $\alpha: G \to \mathrm{Aut}(\mathscr{R})$ is a representation of G by *-automorphisms of $\mathscr{R}$ such that the map $g \in G \to \alpha_g(A) \in \mathscr{R}$ is ultraweakly continuous for every $A \in \mathscr{R}$.

Amply motivated both by mathematical and physical reasons, such non-commutative dynamical systems have been studied for several years by many authors, part of them being interested in particular in the determination of the algebraic type for $\mathscr{R}$ (e.g. [1–9]).

As a motivation example, let us give a direct proof, in the factor case, of a theorem of Hugenholtz [6] and Størmer [2, 7] (the general proof would not be more difficult).

Theorem 1. *Let $\{\mathscr{R}, G, \alpha\}$ be a W^*-system, where $\mathscr{R}$ acts on a Hilbert space $\mathscr{H}$. Let us assume the existence of a unitary representation U of G on $\mathscr{H}$, which implements α, such that $U(g)\xi = \xi$, $g \in G$, where $\xi \in \mathscr{H}$ is cyclic for $\mathscr{R}$ and $\mathbb{C}\xi$ are the only U-invariant vectors. Then $\mathscr{R}$ is of type III or ξ is a trace vector for $\mathscr{R}'$.*

Proof in the factor case. We can assume ξ cyclic and separating for $\mathscr{R}$, otherwise considering $E\mathscr{R}E$, $E \equiv [\mathscr{R}'\xi]$. Let ω be the positive functional $\omega(A) = (A\xi, \xi)$, $A \in \mathscr{R}$, and σ^ω its modular group. We have to show that if $\mathscr{R}$ is semifinite, then ω is a trace, that is σ^ω is trivial. If $\mathscr{R}$ is semifinite there exists a one-parameter unitary group