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On Contraction of Lie Algebra Representations

U. Cattaneo[1]* and W. Wreszinski[2]**

[1] Institut de Physique, Université de Neuchâtel, CH-2000 Neuchâtel, Switzerland
[2] Seminar für theoretische Physik, ETH Hönggerberg, CH-8093 Zürich, Switzerland

Abstract. Given a net $(\mathfrak{g}_\iota)$ of finite-dimensional real Lie algebras contracting into a Lie algebra $\hat{\mathfrak{g}}$, a representation $\hat{\pi}_J$ of $\hat{\mathfrak{g}}$ is constructed explicitly as "limit" of a net (π_ι) of representations, each π_ι being a representation of $\mathfrak{g}_\iota$ on a complex Hilbert space $\mathfrak{H}_\iota$. Conditions are imposed on the net (π_ι) implying that the carrier space of $\hat{\pi}_J$ contain a $\hat{\pi}_J(\hat{\mathfrak{g}})$-stable set of vectors which are analytic for all $\hat{\pi}_J(g)$ $(g \in \mathscr{G})$, where $\mathscr{G}$ is a basis of $\hat{\mathfrak{g}}$. As a corollary, the corresponding result for contractions of representations of simply connected finite-dimensional real Lie groups is derived.

I. Introduction

In this note, we present a theory of contraction of nets of Lie algebra representations. Let J be a directed system (usually a subset of $\mathbf{R}$ with the induced ordering) and, for each $\iota \in J$, let π_ι be a representation of a finite-dimensional real Lie algebra $\mathfrak{g}_\iota$ on a complex Hilbert space $\mathfrak{H}_\iota$. Suppose that, in addition, every $\mathfrak{g}_\iota$ is isomorphic to a reference Lie algebra $\mathfrak{g}$ which is "contracting into $\hat{\mathfrak{g}}$" in a precise sense reviewed in Sect. II.2. We define and investigate a representation $\hat{\pi}_J$ of the contracted Lie algebra $\hat{\mathfrak{g}}$ whose carrier space is constructed in terms of the net $(\mathfrak{H}_\iota)$. The adopted definition permits to give the operators of $\hat{\pi}_J$ directly, without appealing to matrix elements, in a way which seems naturally suited to the problem considered. Since the theory of Lie algebra contraction is rooted in a notion of limit, which is responsible for the fact that $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ are not, in general, isomorphic, the final space cannot be defined in a "canonical" way [for instance, as the Hilbert sum or as a tensor product of the family $(\mathfrak{H}_\iota)$]. Our definition is rather more similar in spirit to Trotter's definition of a Banach space approximated by a sequence of Banach spaces ([1], Sect. 2), with the main difference that Trotter presupposes knowledge of the final space.

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** Present address: Instituto de Física, Universidade de São Paulo, São Paulo, Brazil

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