Commun. math. Phys. 65, 189–201 (1979)

Communications in
Mathematical
Physics

Geometrization of Quantum Mechanics

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Abstract. Quantum mechanics is cast into a classical Hamiltonian form in terms of a symplectic structure, not on the Hilbert space of state-vectors but on the more physically relevant infinite-dimensional manifold of instantaneous pure states. This geometrical structure can accommodate generalizations of quantum mechanics, including the nonlinear relativistic models recently proposed. It is shown that any such generalization satisfying a few physically reasonable conditions would reduce to ordinary quantum mechanics for states that are "near" the vacuum. In particular the origin of complex structure is described.

1. Introduction

Geometrical ideas, especially symplectic structures, have come to play an increasingly important role in classical mechanics [1, 2]. The geometry of the classical phase space Γ also underlies the geometrical quantization programme of Souriau [3, 4] (see also Kostant [5]). Moreover it is known that quantum dynamics can be expressed in terms of a Hamiltonian structure on the Hilbert space $\mathscr{H}$ of state-vectors, where the imaginary part of the scalar product defines a symplectic structure [6].

However if one is seeking an axiomatic basis for quantum mechanics, it seems better to start from structures of direct physical significance, as in the operational approach of Haag and Kastler [7] and others [8, 9] or the work on the geometry of quantum logics [10, 11].

It is pointed out in Sect. 2 that this can be achieved by a slight modification of the Hamiltonian formalism. We have to consider not the Hilbert space $\mathscr{H}$ itself but the manifold Σ of "instantaneous pure states" which is (essentially but not quite) a projective Hilbert space. This formalism is closely akin to the work of Mielnik [12] on the geometry of the space of quantum states. It provides a convenient framework within which to discuss possible generalizations of quantum mechanics. In particular it can readily accommodate the relativistic model theories proposed in a recent paper [13], or at least a large class of them. It is