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Euclidean Nonlinear Classical Field Equations with Unique Vacuum

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Abstract. We study the real, Euclidean, classical field equation

$$(\mu^2 - \Delta)\varphi + \lambda F(\varphi) = f, \qquad \mu^2 > 0,$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is suitably small at infinity. We study existence and regularity assuming that $\lambda \geqq 0$, $F \in C^\infty(\mathbb{R})$, and $aF(a) \geqq 0 \,\forall a \in \mathbb{R}$. These hypotheses allow strongly nonlinear F and nonunique solutions for $f \neq 0$. When $F' \geqq 0$, we prove uniqueness, various contractivity properties, analytic dependence on the coupling constant λ, and differentiability in the external source f. For applications in the loop expansion in quantum field theory, it is useful to know that φ is in the Schwartz class $\mathcal{S}$ whenever f is, and we provide a proof of this fact. The technical innovations of the problem lie in treating the noncompactness of $\mathbb{R}^d$, the strong nonlinearity of F, and the polynomial weights in the seminorms defining $\mathcal{S}$.

I. Introduction

It is well known [1] that the tree approximation to the first functional derivative of the time-ordered, connected generating functional of a boson quantum field theory obeys the classical field equation with an external source. The tree approximation to the connected generating functional is an infinite formal sum in powers of the coupling constant, over Feynman graphs with no loops. It is the zeroth order term in the loop expansion, which is a formal power series expansion in $\hbar c$.

The same correspondence holds in the Euclidean version of boson field theory, where the time-ordered generating functional is replaced by the Laplace transform of the interacting Euclidean measure on $\mathrm{Re}\,\mathcal{S}'$. We refer to the Laplace transform rather than the Fourier transform in order to arrange for a real classical field with the better sign of the coupling constant.

We want to discuss the classical field equation itself in this paper, and not how it emerges from the limit $\hbar c = 0$; but since it was the initial motivation, we describe

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