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Construction of the Affine Lie Algebra $A_1^{(1)}$

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Abstract. We give an explicit construction of the affine Lie algebra $A_1^{(1)}$ as an algebra of differential operators on $\mathbb{C}[x_1, x_3, x_5, \ldots]$. This algebra is spanned by the creation and annihilation operators and by the homogeneous components of a certain "exponential generating function" which is strikingly similar to the vertex operator in the string model.

1. Introduction

For every complex simple Lie algebra $\mathfrak{a}$ there is an infinite-dimensional Lie algebra $\mathfrak{a}\hat{}$ called the associated affine algebra. The affine algebras are among the generalized Cartan matrix (GCM) Lie algebras (or Kac-Moody Lie algebras), which were introduced and studied by Kac [3a] and Moody [7], and which have recently received a great deal of attention. The simplest non-trivial GCM Lie algebra is the affine Lie algebra $\mathfrak{sl}(S, \mathbb{C})\hat{}$. [Here $\mathfrak{sl}(2, \mathbb{C})$ denotes the Lie algebra of traceless 2 by 2 complex matrices. The Kac-Moody definition of $\mathfrak{sl}(2, \mathbb{C})\hat{}$ is given in §2 below.] This algebra is denoted $A_1^{(1)}$ by Kac. For convenience we will henceforth write $\mathfrak{g}$ for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}\hat{}$ for $\mathfrak{sl}(2, \mathbb{C})\hat{}$.

The main purpose of this paper is to construct $\mathfrak{g}\hat{}$ as a concrete Lie algebra of differential operators on the space $\mathbb{C}[x_1, x_3, x_5, \ldots]$ of polynomials in infinitely many variables. (This space is naturally graded by setting $\deg x_k = -k$.) In this construction (described in detail in § 5) $\mathfrak{g}\hat{}$ is spanned by the identity, the creation and annihilation operators [$L(x_k)$ and $\partial/\partial x_k$], and the homogeneous components of

$$\exp(\sum 4L(x_k)/k)\exp(-\sum \partial/\partial x_k)\ .$$

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