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Proof of an Entropy Conjecture of Wehrl

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Abstract. Wehrl has proposed a new definition of classical entropy, S, in terms of coherent states and conjectured that $S \geqq 1$. A proof of this is given. We discuss the analogous problem for Bloch coherent spin states, but in this case the conjecture is still open. An inequality for the entropy of convolutions is also given.

I. Introduction

In a recent paper [1], A. Wehrl introduced a new definition of the "classical" entropy corresponding to a quantum system, proved that it had several interesting properties that deserve to be studied further, and posed a conjecture about the minimum value of this "classical" entropy. The main purpose of this paper is to prove Wehrl's conjecture. It is somewhat surprising that while the conjecture appears to be almost obvious, the proof we give requires some difficult theorems in Fourier analysis. The conjecture may or may not be important physically, but it reveals an interesting feature of coherent states.

To briefly recapitulate Wehrl's analysis, consider a single particle in one dimension, so that the Hilbert space is $L^2(\mathbb{R})$. (The generalization to $\mathbb{R}^N$ is trivial.) For each $z=(p,q)\in\mathbb{R}^2$, define the normalized vector $|z\rangle$ in $L^2(\mathbb{R})$ by

$$|z\rangle \equiv (\pi\hbar)^{-1/4}\exp\{[-(x-q)^2/2+ipx]/\hbar\} \equiv R(x|p,q). \tag{1.1}$$

These vectors are the coherent states used by Schrödinger [2], Bargmann [3], Klauder [4], and Glauber [5]. If

$$P_z = |z\rangle\langle z| \tag{1.2}$$

is the orthogonal projection onto $|z\rangle$ then

$$\int \frac{dz}{\pi} P_z = I\,, \tag{1.3}$$

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