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The φ_2^4 Quantum Field as a Limit of Sine-Gordon Fields

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Abstract. We exhibit the $\lambda\varphi_2^4$ quantum field theory as the limit of Sine-Gordon fields as suggested by the identity

$$\varphi^4/4! = \lim_{\varepsilon\to 0}(\varepsilon^{-4}\cos\varepsilon\varphi - \varepsilon^{-4} + \tfrac{1}{2}\varepsilon^{-2}\varphi^2).$$

The proofs of finite volume stability for the two models, due to Nelson and Fröhlich respectively, are unrelated. We find a generalized stability argument that incorporates ideas from both of the simpler cases. The above limit, for the Schwinger functions, then proceeds uniformly in ε.

As a by-product, let $(\varphi, d\mu)$ be a Gaussian random field, φ_κ $(1 \leqq \kappa < \infty)$ a regularization of φ, and V a function satisfying:

(i) $V(\varphi_\kappa) \geqq -a\kappa^\alpha$

(ii) $\| V(\varphi) - V(\varphi_\kappa)\|_p \leqq bp^\beta\kappa^{-\gamma}, \qquad 2 \leqq p < \infty.$

Then $e^{-V(\varphi)} \in L^1(d\mu)$ provided $\alpha(\beta - 1) < \gamma$.

I. Introduction and Results

In this paper we show how to obtain the $\lambda\varphi_2^4$ quantum field theory as a uniform limit of Sine-Gordon ($\lambda_s \cos\varepsilon\varphi$) quantum fields. Formally one might expect such a relationship as a consequence of the identity

$$\lambda\varphi^4/4! = \lim_{\varepsilon\to 0}\lambda(\varepsilon^{-4}\cos\varepsilon\varphi - \varepsilon^{-4} + \tfrac{1}{2}\varepsilon^{-2}\varphi^2), \tag{1.1}$$

which suggests convergence of the $\lambda_s \cos\varepsilon\varphi$ model to $\lambda\varphi^4$ as $\varepsilon \to 0$, provided we perform infinite vacuum energy, mass and coupling constant renormalizations.

There are serious technical problems to be overcome before this idea may be extended to quantum field theory. To prove convergence of the corresponding Schwinger functions, some uniformity in ε will be needed, such as a uniform bound for $\langle e^{-V_\varepsilon}\rangle$ where V_ε is the finite-volume action. However the proofs that e^{-V} is integrable for the φ_2^4 and Sine-Gordon theory, due respectively to Nelson [1]

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