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Unbounded Derivations of Commutative C^*-Algebras

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Abstract. It is shown that an unbounded *-derivation δ of a unital commutative C^*-algebra A is quasi well-behaved if and only if there is a dense open subset U of the spectrum of A such that, for any f in the domain of δ, $\delta(f)$ vanishes at any point of U where f attains its norm. An example is given to show that even if δ is closed it need not be quasi well-behaved. This answers negatively a question posed by Sakai for arbitrary C^*-algebras.

It is also shown that there are no-zero closed derivations on A if the spectrum of A contains a dense open totally disconnected subset.

1. Introduction

Unbounded derivations have recently become one of the most important branches of the theory of C^*-algebras, since they include the infinitesimal generators of the one-parameter *-automorphism groups representing time-evolution of quantum dynamical systems. Several authors have shown how results from Banach space theory take on special forms for C^*-algebras (see e.g. [2, 3]). In his recent survey of the theory of unbounded derivations [6], Sakai raised several questions concerning closed *-derivations. In this paper, we obtain negative answers to two of these questions.

Sakai proved that a sufficient condition for the commutative C^*-algebra $C(\Omega)$ of continuous complex-valued functions on a compact Hausdorff space Ω to have no non-zero closed *-derivations is that Ω should be totally disconnected, and asked whether this condition is also necessary [6, Problem 1.1]. We show here that it is not by proving that another weaker sufficient condition is that Ω should contain a dense open totally disconnected subset. This result has also been obtained independently by B. E. Johnson.

Let δ be a *-derivation of any C^*-algebra A [δ is assumed to have dense domain $\mathscr{D}(\delta)$]. An element x of the self-adjoint part $\mathscr{D}(\delta)^s$ of $\mathscr{D}(\delta)$ is said to be *well-behaved* if there is a state ϕ of A with $|\phi(x)|=\|x\|$ and $\phi(\delta(x))=0$, and to be *strongly well-behaved* if $\phi(\delta(x))=0$ for all self-adjoint linear functionals ϕ in A^* with