Commun. math. Phys. 161, 191—208 (1978)

Communications in Mathematical Physics

Correlation-Function Identities and Inequalities for Ising Models with Pair Interactions

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Abstract. For Ising models with pair interactions in zero magnetic field a class of linear combinations of products of two correlation functions is studied. We derive sufficient and necessary conditions under which a function in this class is (a) zero for all values of the coupling parameters, or (b) nonnegative for all nonnegative values of the coupling parameters. Examples of correlation-function identities and inequalities of this type are given.

1. Introduction

In a recent paper [1], to be referred to as I, it was proved that for a (zero-field) Ising model on a planar graph the correlation functions for spins on a so-called boundary set satisfy certain algebraic relations which are valid for all values of the coupling parameters between the spins. These relations can all be derived from a set of identities which can be written in the form

$$\sum_{j=1}^{n} (-1)^j \langle \sigma_{v_1} \sigma_{v_j} \rangle \left\langle \sigma_{v_1} \sigma_{v_j} \prod_{k=1}^{n} \sigma_{v_k} \right\rangle = 0,$$

where $(v_1, \ldots, v_n)$ is a sequence of (not necessarily distinct) vertices which is a boundary sequence of the planar graph [cf. I Eq. (17)].

The left-hand side of the above equation can be considered as a special case of a function of the type $\sum_{B \subset A} \lambda_B \langle \sigma_B \rangle_G \langle \sigma_B \sigma_D \rangle_G$, where G is an arbitrary graph, A is an arbitrary set of vertices of G, D a subset of A, the sum is over all subsets of A, and the coefficients λ_B are independent of the coupling parameters; the case where all vertices in the boundary sequence are different then corresponds to the case $D = A$, i.e. $\sigma_B \sigma_D = \sigma_{A \setminus B}$.

In this paper we derive sufficient and necessary conditions on the coefficients λ_B under which a function of this general type for an Ising model with pair interactions in zero magnetic field is (a) zero for all values of the coupling parameters, or (b) nonnegative for all nonnegative values of the coupling parameters, respectively.

0010-3616/78/0061/0191/$03.60