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Local Decay of Scattering Solutions to Schrödinger's Equation

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Abstract. The main theorem asserts that if $H=\Delta+gV$ is a Schrödinger Hamiltonian with short range V, $\phi\in L^2_{\text{compact}}$ $(\mathbb{R}^3)$, and $R>0$, then $\|\exp(iHt)\Pi_S\phi\|_{L^2(|x|<R)}=O(t^{-1/2})$ as $t\to\infty$ where Π_S is projection onto the orthogonal complement of the real eigenvectors of H. For all but a discrete set of g, $O(t^{-1/2})$ may be replaced by $O(t^{-3/2})$.

§ 1. Introduction

A basic dynamical equation in nonrelativistic quantum mechanics is Schrödinger's equations for $u(t,x)$:

$$\frac{1}{i}\frac{\partial u}{\partial t}=\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}+V(x)u \tag{1.1}$$

where $t\in\mathbb{R}$ and $x\in\mathbb{R}^3$. Let $H_0\equiv\Delta\equiv\sum\frac{\partial^2}{\partial x_i^2}$ and $H\equiv\Delta+V$. The formal solution of (1.1) is $u(t)=e^{iHt}u(0)$. If V is real valued and not too singular too example if $V\in L_2(\mathbb{R}^3)$), then H with domain the Sobolev space $W_2(\mathbb{R}^3)$ is selfadjoint on $L_2(\mathbb{R}^3)$ so e^{iHt} defines a one parameter group of unitary maps. Two types of solution are easily visualized: bound states, $e^{i\omega t}\phi(x)$, for which $|u(t,x)|$ is independent of time, and, scattering solutions $u(t,x)$ with the property that for any ball $\mathscr{B}$ in $\mathbb{R}^3$

$$\int_{\mathscr{B}}|u(t,x)|^2\,dx\to 0 \quad \text{as} \quad t\to\infty\,. \tag{1.2}$$

If $V(x)\to 0$ as $|x|\to\infty$, it is natural to expect that for solutions in the latter class there is a $u_+(t)=e^{itH_0}u_+(0)$ with $\|u(t)-u_+(t)\|_{L^2(\mathbb{R}^3)}\to 0$ as $t\to\infty$. There is a large and rich literature devoted to showing that these two types of solution form an exhaustive list, that is, $L_2(\mathbb{R}^3)$ can be written as an orthogonal direct sum $\mathscr{H}_B\oplus\mathscr{H}_S$ such that e^{itH} maps both $\mathscr{H}_B$ and $\mathscr{H}_S$ into themselves, $\mathscr{H}_B$ is spanned by eigenvectors of H, and $\mathscr{H}_S$ consists of solutions which decay locally in the sense of (1.2). For details see ([1, 5, 6, 13]).

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