Commun. math. Phys. 61, 97—118 (1978)

Communications in
Mathematical
Physics

Topological Aspects of Yang-Mills Theory

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Abstract. The space of maps $S^3 \to G$ has components which give the topological quantum number of Yang-Mills theory for the group G. Each component of the space has further topological invariants. When $G = \mathrm{SU}(2)$ we show that these invariants (the homology groups) are "captured" by the space of instantons. Using these invariants we show that potentials must exist for which the massless Dirac equation (in Euclidean 4-space) has arbitrarily many independent solutions (for fixed instanton number).

§1. Introduction

In non-abelian 4-dimensional gauge theories it is by now well-known that certain topological aspects play an important role. More specifically the fact that, for a simple non-abelian compact Lie group G, the third homotopy group $\pi_3(G)$ is isomorphic to the integers leads to a "topological quantum number" k. The purpose of this paper is to draw attention to further topological features and to show how these are related to analytical aspects of the gauge theory.

Our basic observation is that the Euclidean Yang-Mills Lagrangian is defined on a function space with many components (labelled by the integers) and that each component has further internal topological invariants. Homotopically the function space is determined by the asymptotic data and so it can be identified with the space $\Omega^3(G)$ of maps $S^3 \to G$ (normalized to preserve base points). The components of this space give $\pi_3(G)$ and are labelled by integers k, and each component $\Omega^3_k(G)$ is a space with much internal structure extensively studied by topologists.

As usual in order to deal with the asymptotic conditions in R^4 we shall work on the 4-sphere $S^4 = R^4 \cup \infty$ which is the conformal compactification. In this version the relevant function space is the space of connections (potentials) modulo gauge transformations. The space of connections is a linear space, with no topological invariants, but after factoring out by gauge transformations we get a space $\mathscr{C}(G)$ which is homotopically $\Omega^3(G)$. These basic facts are described in §2.

0010-3616/78/0061/0097/$04.40