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The Unified Approach to Spectral Analysis

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Abstract. We develop a new, unified, method to construct a closed (selfadjoint in $\mathscr{L}^2$) extension of a partial differential operator in all the spaces $\mathscr{L}^p(\mathbb{R}^n)$ $1 \leqq p \leqq \infty$. Our method is not only an unified approach but it is also very efficient. We obtain very weak conditions on the potentials.

I. Introduction

In this paper we develop a new method to construct a closed extension (selfadjoint in the case $\mathscr{L}^2$) of a partial differential operator in $\mathscr{L}^p(R^n)$, $1 \leqq p \leqq \infty$, and to study its spectral properties.

Let P_0 be a constant coefficients partial differential operator and let $Q = \sum_{i=1}^{M} q_i Q_i$ be a lower order variable coefficients partial differential operator. The basic object of our method is the formal series expansion for the resolvent of the perturbed operator $P_0 + Q$:

$$R(z) = \sum_{n=0}^{\infty} R_0(z)(QR_0(z))^n.$$

$R_0(z)$ being the resolvent of the unperturbed operator P_0.

Our strategy is to prove, by an interpolation argument, that the formal series expansion defines a bounded operator in all the $\mathscr{L}^p(\mathbb{R}^n)$ spaces, $1 \leqq p \leqq \infty$, denoted $R_p(z)$. Then we prove that $R_p(z)$ is the resolvent of a closed extension of the perturbed operator $P_0 + Q$.

Several methods have been proposed to construct a closed extension (selfadjoint in $\mathscr{L}^2$) of a partial differential operator in $\mathscr{L}^p$, $1 \leqq p \leqq \infty$. In $\mathscr{L}^p$, $1 \leqq p < \infty$, if the potentials $q_i(x)$ are locally in $\mathscr{L}^p$, the closed extension is defined as the operator sum of P_0 and Q. For a general treatment of this method see [3] were references to original contributions are given.

The method of quadratic forms extensions in $\mathscr{L}^2$ has been developed by [2] (and the references to original contributions quoted there) [5], [6], [7], and [8]. In $\mathscr{L}^p$, $1 < p < \infty$, the method of S-extensions has been created by M. Schechter [3].

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