Commun. math. Phys. 59, 273—283 (1978)

Communications in
Mathematical Physics

Decomposable Differential Operators in a Cosmological Context

R. Maartens and S. D. Nel
Department of Applied Mathematics, University of Cape Town, Rondebosch, Cape Town, South Africa

Abstract. The integrability conditions for a certain second order ordinary differential equation in two variables are studied via the concept of decomposability of the associated differential operator. The results are applied to regain in a unified manner the known exact solutions for locally rotationally symmetric, spatially homogeneous cosmological models. In addition, new solutions are obtained.

1. Introduction

Einstein's equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab} \tag{1.1}$$

for a spacetime metric tensor g_{ab}, where R_{ab} is the Ricci tensor, $R = R^a_a$, and Λ is the cosmological constant, can be solved exactly only in cases of rather high spacetime symmetry, and for relatively simple forms of the energy-momentum tensor T_{ab}.

In this paper, we consider exact solutions of (1.1) for spacetimes in which local coordinates $(x^a) = (t, x^\alpha)(a = 0, \ldots, 3; \alpha = 1, \ldots, 3)$ may be chosen so that one or more of the field equations, or combinations thereof, take the generic form

$$A_1 \frac{\ddot{X}}{X} + A_2 \frac{\ddot{Y}}{Y} + A_3 \frac{\dot{X}^2}{X^2} + A_4 \frac{\dot{Y}^2}{Y^2} + A_5 \frac{\dot{X}}{X} + A_6 \frac{\dot{Y}}{Y} + A_7 \frac{\dot{X}\dot{Y}}{XY} + H(X, Y, t) = 0 \tag{1.2}$$

where $A_i \in \mathbf{R}(i = 1, \ldots, 7)$, a dot denotes differentiation with respect to t, and $X(t)$, $Y(t)$ are metric component functions. This is the case, for example, when the spacetime is locally rotationally symmetric and admits a 1-parameter family of homogeneous hypersurfaces (see, e.g., Refs. [1–3]). If a first integral of (1.2) can be found, then together with the remaining field equations and the conservation equations $T^{ab}_{;b} = 0$, this often allows us to obtain a reduction of the system of equations to quadratures, or to a single ordinary differential equation in one variable, plus quadratures.

0010-3616/78/0059/0273/$02.20