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Stable Vector Bundles and Instantons*

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Abstract. Methods of abstract algebraic geometry are used to study rank 2 stable vector bundles on $\mathbb{P}^3$. These bundles are then used to give self-dual solutions, called instantons, of the Yang-Mills equation on S^4.

§0. Introduction

A problem which has recently attracted considerable attention in the gauge theories of elementary particle physics is the search for self-dual solutions of the Yang-Mills equation. The solutions found so far depend on a finite set of points of the 4-sphere S^4, and are called instantons. In the language of differential geometry, this problem can be phrased as follows: find all possible connections with self-dual curvature on a smooth SU(2)-bundle over the 4-sphere S^4. Following the Penrose program of translating physical problems into problems of several complex variables, Atiyah and Ward [2] have shown that this problem is equivalent to the classification of certain holomorphic $\mathbb{C}^2$-bundles on complex projective 3-space $\mathbb{P}^3_{\mathbb{C}}$ with an added reality condition (described explicitly in §1).

An instanton has a "topological quantum number" k, which is a positive integer. The physicists [7] have already shown the existence of instantons for all values of $k=1,2,\ldots$. In the differential-geometric interpretation, k is the second Chern class c_2 of the SU(2)-bundle; it is known that c_2 completely determines a smooth SU(2)-bundle on S^4 up to isomorphism. Atiyah et al. [1] have shown that the moduli (or parameter space) for the set of connections with self-dual curvature on the SU(2)-bundle with given $c_2>0$ is a smooth manifold of dimension $8c_2-3$. Their proof, which uses deformation theory and the index theorem, is purely local and tells nothing of the global properties of the moduli space. To find out more about this moduli space, in particular, whether it is connected, and to give an explicit construction for the corresponding instantons, it seems that recent work in

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