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Action Minima among Solutions to a Class of Euclidean Scalar Field Equations

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Abstract. We show that for a wide class of Euclidean scalar field equations, there exist non-trivial solutions, and the non-trivial solution of lowest action is spherically symmetric. This fills a gap in a recent analysis of vacuum decay by one of us.

1. Introduction

In the course of a study of vacuum instability [1], one of us encountered the differential equation in four-dimensional Euclidean space,

$$\Delta\Phi = U'(\Phi). \tag{1.1}$$

Here Δ is the usual Euclidean Laplace operator, U is a quartic polynomial in the single real field Φ, and the prime denotes differentiation with respect to Φ. This equation admitted a trivial solution, Φ a constant. In Ref. [1], a spherically symmetric non-trivial solution was constructed, and it was conjectured that this solution had the lowest action of any non-trivial solution. The purpose of this note is to supply the proof of this conjecture.

More precisely, we prove that, for a wide class of functions U, the non-trivial solution to Equation (1.1) of smallest action is necessarily spherically symmetric. Our proof is valid for any number of Euclidean dimensions greater than two, although the class of admissible U's does depend upon the dimension[1].

The remainder of this section is a statement of our main result with some comments on its meaning. Sections 2 and 3 consist of the proof.

1.1. Statement of the Theorem

Definition. We will say a real function of a single real variable $U(\Phi)$ is admissible

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1 In particular, our theorem applies to the non-polynomial U considered by Frampton [2]

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