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Integrability of the Classical $[\bar{\psi}_i\psi_i]_2^2$ and $[\bar{\psi}_i\psi_i]_2^2 - [\bar{\psi}_i\gamma_5\psi_i]_2^2$ Interactions*

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Abstract. We study the interaction of N classical two-dimensional massless Fermi fields through the symmetric couplings $[\bar{\psi}_i\psi_i]^2$ or $[\bar{\psi}_i\psi_i]^2 - [\bar{\psi}_i\gamma_5\psi_i]^2$. We explicitly show complete integrability in the cases $N = 1, 2$, using the inverse scattering method. The fields occuring in the associated linear eigenvalue problem and evolution equation are simply related to the fundamental fields ψ_i that satisfy the original non-linear equations. For $N > 2$, calculations become very involved, but there is no doubt that the system remains completely integrable, reducing to appropriate generalizations of the sin$e$- and sin$h$-Gordon equation, a situation analogous to Pohlmeyer's discussion in a somewhat similar problem: the two-dimensional non-linear σ-model. Finally, all the explicit analytic solutions that we have worked out in the present framework are identical to those found by Dashen et al., and Shei, in a semiclassical treatment of the fully quantum mechanical version of these models. This leads us to conjecture that the quantum theory also shares most of the features of completely integrable systems, like the massive Thirring model.

I. Introduction

The theory of massless fermions with scalar contact interactions, first introduced by Nambu and Jona-Lasinio [1] as a model field theory for superconductors, is renormalizable in two space-time dimensions. It has been studied in [2], in the limit where the number N of fermion species goes to infinity, and was shown to be asymptotically free and exhibit dynamical spontaneous symmetry breaking. In the spirit of the so-called $1/N$ expansion, the above theory was further analyzed in [3] by partially integrating out the Fermi fields, reducing to an effective Lagrangian for the scalar composite field $\sigma = \bar{\psi}\psi$. To leading order in the adopted approximation scheme, the effective Lagrangian for the field σ was studied by semiclassical

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