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Simple C^*-Algebras Generated by Isometries

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Abstract. We consider the C^*-algebra $\mathcal{O}_n$ generated by $n \geqq 2$ isometries $S_1, \ldots, S_n$ on an infinite-dimensional Hilbert space, with the property that $S_1 S_1^* + \ldots + S_n S_n^* = \mathbf{1}$. It turns out that $\mathcal{O}_n$ has the structure of a crossed product of a finite simple C^*-algebra $\mathcal{F}$ by a single endomorphism scaling the trace of $\mathcal{F}$ by $1/n$. Thus, $\mathcal{O}_n$ is a separable C^*-algebra sharing many of the properties of a factor of type III_λ with $\lambda = 1/n$. As a consequence we show that $\mathcal{O}_n$ is simple and that its isomorphism type does not depend on the choice of $S_1, \ldots, S_n$.

A C^*-algebra is simple if it contains no non-trivial closed two-sided ideals. We call a simple C^*-algebra with unit infinite if it contains an element X such that $X^*X = \mathbf{1}$ and $XX^* \neq \mathbf{1}$. While non-separable algebras of this type are well known (e.g. the Calkin algebra or type III factors on a separable Hilbert space) there is to my knowledge no explicit example of a separable simple infinite C^*-algebra. The existence of such algebras was proved by Dixmier in [9, 2.1] by the following argument. Let S_1, S_2 be two isometries ($S_i^* S_i = \mathbf{1}, i = 1, 2$) on an infinite-dimensional Hilbert space $\mathcal{H}$ such that $S_1 S_1^* + S_2 S_2^* = \mathbf{1}$. Since the C^*-algebra $C^*(S_1, S_2)$ generated by S_1 and S_2 has a unit, it contains a maximal proper two-sided ideal $\mathcal{J}$. The quotient $C^*(S_1, S_2)/\mathcal{J}$ is separable, simple and infinite. One of the results of the present paper is that $C^*(S_1, S_2)$ itself is already simple (thus answering the question of Dixmier to this effect). More generally, we study the C^*-algebra generated by $n \geqq 2$ isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^{n} S_i S_i^* = \mathbf{1}$ (this condition implies in particular that the range projections $S_i S_i^*$ are pairwise orthogonal). We include the case $n = \infty$. We note incidentally that J. Roberts, motivated by investigations on superselection sectors, has studied closed linear spaces generated by isometries with this property [15]. These spaces are in fact Hilbert spaces and $C^*(S_1, \ldots, S_n)$ is from this point of view the C^*-algebra generated by a Hilbert space.

We construct a faithful conditional expectation of $C^*(S_1, \ldots, S_n)$ onto a C^*-subalgebra $\mathcal{F}$ and show that $C^*(S_1, \ldots, S_n)$ is the crossed product of $\mathcal{F}$ by a single endomorphism Φ (in a sense to be made precise in Section 2). If n is finite, then $\mathcal{F}$ is a