Commun. math. Phys. 55, 149—162 (1977)

Communications in Mathematical Physics
© by Springer-Verlag 1977

Existence of Solitary Waves in Higher Dimensions*

Walter A. Strauss

Department of Mathematics, Brown University, Providence, Rhode Island 02912, USA

Abstract. The elliptic equation $\Delta u = F(u)$ possesses non-trivial solutions in $\boldsymbol{R}^n$ which are exponentially small at infinity, for a large class of functions F. Each of them provides a solitary wave of the nonlinear Klein-Gordon equation.

1. Introduction

We define a solitary wave as a solution $\phi(x, t)$ of a wave equation whose maximum amplitude at time t, $\sup_x |\phi(x, t)|$, does not tend to zero as $t \to \infty$, but which tends to zero in some convenient sense as $|x| \to \infty$ for each t. The convergence should have the property that physical quantities, such as the energy and charge, are finite. Particular types of solitary waves are (1) traveling waves $\phi = u(x - ct)$ where c is a constant vector and (2) standing waves $\phi = \exp(i\omega t)u(x)$ where ω is a real constant. Traditionally, solitary waves have been traveling waves, but in recent years oscillatory factors have been allowed. The above definition includes all uses of the term. Solitary waves have also been called "solitons" but, properly speaking, the latter word should be reserved for those special solitary waves which exactly preserve their shapes after interaction. Many examples of these special solitons have been discovered in recent years in the case of two space-time dimensions. In higher dimensions, however, even the existence of solitary waves seems to be elusive.

We consider the scalar NLKG equation

$$\phi_{tt} - \Delta\phi + m^2\phi + f(\phi) = 0 , \tag{3}$$

where $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$, Δ is the Laplacian in x and $m > 0$. We assume $f(0) = 0$ and $f(re^{i\theta}) = f(r)e^{i\theta}$. If ϕ is a standing wave (2), Equation (3) reduces to

$$-\Delta u + (m^2 - \omega^2)u + f(u) = 0 . \tag{4}$$

We shall show that (4) possesses non-trivial solutions exponentially small at infinity provided $|\omega| < m$ and f satisfies certain conditions. In particular, if $\omega = 0$, we have $\phi(x) = u(x)$. Since we may change to a different Lorentz frame, it follows that there exist traveling solitary waves (1) for any $|c| < 1$. Alternatively we may proceed by

* This work was supported in part by NSF Grant MCS 75-08827