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Instantons and Algebraic Geometry

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Abstract. Minimum action solutions for SU(2) Yang-Mills fields in Euclidean 4-space correspond, via the Penrose twistor transform, to algebraic bundles on the complex projective 3-space. These bundles in turn correspond to algebraic curves. The implication of these results for the Yang-Mills fields is described. In particular all solutions are rational and can be constructed from a series of Ansätze A_l for $l \geqq 1$.

§ 1. Introduction

The term instanton or pseudo-particle has been coined for the minimum action solutions of SU(2) Yang-Mills fields in Euclidean 4-space R^4. Conditions at infinity are imposed which are tantamount to working on the 4-sphere S^4 (which is the conformal compactification of R^4) and are classified by an integer k, which is interpreted as the "number of instantons". The most general solutions so far constructed explicitly are those of Jackiw et al. [4]. They depend on $5|k|+4$ real parameters if $|k| \geqq 3$, while for $|k|=1,2$ the number of parameters are 5, 13 respectively. On the other hand infinitesimal deformation theory shows that the number of parameters for the complete family of solutions is $8|k|-3$ (see [1, 5, 9]). The purpose of this note is to describe how the full $(8|k|-3)$-parameter family can in principle be constructed by using algebraic geometry.

The basic idea is to use the Penrose Twistor approach to space-time [8] in which field equations in 4-space are converted into complex analytic geometry on complex projective 3-space P_3. This approach was applied in [11] to the self-dual (or anti-self dual) Yang-Mills equations (which correspond to minimum action). The resulting geometrical objects on P_3 turn out to be complex analytic bundles. This transformation can be applied both locally and globally and for either Minkowski or Euclidean space. For the instanton problem we take the global Euclidean version (i.e. for S^4) and this leads to a complex analytic bundle defined over the *whole* of P_3. By Serre's basic theorems of analytic geometry [10] such bundles are necessarily *algebraic*. Moreover algebraic geometers have recently made significant progress on the study of precisely those algebraic bundles which correspond to the SU(2) Yang-