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All Unitary Ray Representations of the Conformal Group SU(2,2) with Positive Energy

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Abstract. We find all those unitary irreducible representations of the ∞-sheeted covering group $\tilde{G}$ of the conformal group $SU(2,2)/\mathbb{Z}_4$ which have positive energy $P^0 \geqq 0$. They are all finite component field representations and are labelled by dimension d and a finite dimensional irreducible representation (j_1, j_2) of the Lorentz group SL(2$\mathbb{C}$). They all decompose into a finite number of unitary irreducible representations of the Poincaré subgroup with dilations.

1. Summary and Introduction

The conformal group of 4-dimensional space time is locally isomorphic to $G = SU(2,2)$; its universal covering group $\tilde{G}$ is an infinite sheeted covering of G. Both G and $\tilde{G}$ contain the quantum mechanical Poincaré group ISL(2$\mathbb{C}$). It is of physical interest to have a complete list of all unitary irreducible representations (UIR's) of $\tilde{G}$ with positive energy $P^0 \geqq 0$. They are at the same time unitary ray representations of G. In the present paper we shall give such a complete list. We show that all the UIR of $\tilde{G}$ with positive energy are finite component field representations in the terminology of [1]. They are labelled by a real number d, called the dimension, and a finite dimensional irreducible representation (j_1, j_2) of the quantum mechanical (q.m.) Lorentz group SL(2$\mathbb{C}$). Thus, $2j_1, 2j_2$ are non-negative integers. There are 5 classes of representations. They differ in their Poincaré content $[m, s]$, m = mass, s = spin resp. helicity as follows:

(1) trivial 1-dimensional representation $d = j_1 = j_2 = 0$.
(2) $j_1 \neq 0, j_2 \neq 0, d > j_1 + j_2 + 2$ contains $m > 0$, $s = |j_1 - j_2| \ldots j_1 + j_2$ (integer steps)
(3) $j_1 j_2 = 0$, $d > j_1 + j_2 + 1$ contains $m > 0$, $s = j_1 + j_2$.
(4) $j_1 \neq 0, j_2 \neq 0, d = j_1 + j_2 + 2$ contains $m > 0$, $s = j_1 + j_2$.
(5) $j_1 j_2 = 0$, $d = j_1 + j_2 + 1$ contains $m = 0$, helicity $j_1 - j_2$.

The proof of these results proceeds in several steps.

We start from the observation [2, 3] that positive energy $P^0 \geqq 0$ implies that also $H \geqq 0$, where $H = \frac{1}{2}(P^0 + K^0)$ is the "conformal Hamiltonian", K^0 a generator of special conformal transformations. Next we point out that any UIR of $\tilde{G}$ with