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A Symplectic Structure on the Set of Einstein Metrics

A Canonical Formalism for General Relativity

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Abstract. A symplectic structure i.e. a symplectic form Γ on the set of all solutions of the Einstein equations on a given 4-dimensional manifold is defined. A degeneracy distribution of Γ is investigated and its connection with an action of the diffeomorphism group is established. A multiphase formulation of General Relativity is presented. A superphase space for General Relativity is proposed.

1. Introduction

It is known that the Hamilton formulation of mechanics is an appropriate tool for the quantization of classical systems. In the sixties this formulation was elegantly presented in a general theory of symplectic manifolds cf. [1, 22]. A basic concept in that approach is a $2n$-dimensional manifold $\mathscr{P}_{2n}$ — a phase space of a dynamical system and a non-degenerate closed 2 form $\overset{2}{\gamma}$ on $\mathscr{P}_{2n}$. The differential form $\overset{2}{\gamma}$ defines a bilinear skewsymmetric form $\{\cdot, \cdot\}$ on the vector space $\mathscr{F}$ of all smooth functions on $\mathscr{P}_{2n}$. The form $\{\cdot, \cdot\}$ is called a Poisson bracket. It defines a Lie algebra structure on the set $\mathscr{F}$. Very often $\mathscr{P}_{2n}$ is the cotangent bundle to an n-dimensional manifold V (a configuration space of a system). Then $\overset{2}{\gamma}$ is the canonical differential 2-form on T^*V and if (q^i) are local coordinates in V, (p_j, q^i) are local coordinates in $\mathscr{P}_{2n} = T^*V$ then

$$\overset{2}{\gamma} = \sum_{i=1}^{n} dp_i \wedge dq^i \tag{1.1}$$

and for $f_1, f_2 \in \mathscr{F} = C^\infty(\mathscr{P}_{2n})$

$$\{f_1, f_2\} = \sum_{i=1}^{n} ((\partial f_1/\partial p_i)(\partial f_2/\partial q^i) - (\partial f_2/\partial p_i)(\partial f_1/\partial q^i)) . \tag{1.2}$$

In recent years was found a generalization of the notion of the symplectic manifold which is useful in classical field theories [15–17, 23]. This construction is based on a geometric theory of the calculus of variations formulated by Dedecker