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Algebraic Implications of Composability of Physical Systems

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Abstract. From classical and quantum mechanics we abstract the concept of a two-product algebra. One of its products is left unspecified; the other is a Lie product and a derivation with respect to the first. From composition of physical systems we abstract the concept of composition classes of such two-product algebras, each class being a semigroup with a unit. We show that the requirement of mutual consistency of the algebraic and the semigroup structures completely determines both the composition classes and the two-product algebras they consist of. The solutions are labelled by a single parameter which in the physical case is proportional to the square of the quantum of action.

I. Introduction

In both classical and quantum mechanics, the set of physical variables belonging to a system with a given number of degrees of freedom is a linear space with *two* algebraic products. The classical variables are real-valued functions on a phase space, and the two products are the usual multiplication and the Poisson bracket of such functions. The quantal variables are self-adjoint operators on a Hilbert space, and the two products are $\frac{1}{2}[\,,]_+$ and $(i\hbar)^{-1}[\,,]_-$, where $[\,,]_+$ and $[\,,]_-$ denote the anticommutator and the commutator of such operators. Multiplication of classical functions is commutative and associative, while the anticommutator of quantal operators is a commutative but not associative product. Both the Poisson bracket and the commutator are Lie products. Further, in both mechanics the two products are related by the same "distribution law", i.e. the derivation rule. For example, if f, g, h are classical variables and $\cdot$ and $\{\,,\}$ denote the two classical products, then $\{f, g\cdot h\} = \{f, g\}\cdot h + g\cdot\{f, h\}$.

From the examples provided by classical and quantum mechanics we abstract the concept of a two-product algebra $\{\mathscr{H}, \tau, \alpha\}$. In the definition of this structure, the properties of the product τ are left unspecified, while the product α is required to be a Lie product and the operators $f\alpha$, with $f \in \mathscr{H}$, are to be derivations with respect to the product τ.