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Conformal Changes and Geodesic Completeness

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Abstract. Let (M, g) be a causal spacetime. Condition N will be satisfied if for each compact subset K of M there is no future inextendible nonspacelike curve which is totally future imprisoned in K. If M satisfies condition N, then whenever E is an open and relatively compact subset of M the spacetime E with the metric g restricted to E is stably causal. Furthermore, there is a conformal factor Ω such that $(M, \Omega^2 g)$ is both null and timelike geodesically complete. If M is an open subset of two dimensional Minkowskian space, then M is conformal to a geodesically complete spacetime.

1. Introduction

Causality is playing an important role in the study of relativity. It is useful in the investigation of black holes [1, 2] and in cosmology [3, 4].

A spacetime which remains causal under slight perturbations of the metric is called stably causal. This is one of the most reasonable causality conditions since quantum effects imply that measurements are always imprecise.

It is an interesting fact that many spacetimes which are not themselves stably causal have relatively large subsets which are stably causal. The question arises of how to decide if a given subset of a causal spacetime is stably causal. The first interesting result of this paper is that if (M, g) satisfies the nonimprisonment condition N, then any open subset E of M with compact closure $\bar{E}$ is stably causal. This means that when condition N is valid and (M, g) is not stably causal the stability condition breaks down near the boundary of M. A corollary of our first theorem is that if M satisfies condition N and μ is a bounded measure on M, then whenever $\varepsilon > 0$ is given there is a closed subset F of M such that $\mu(F) < \varepsilon$ and $M - F$ is stably causal.

Let Ω be a positive real valued function on M. The two metrics g and $\Omega^2 g$ on M are said to be conformally equivalent. The function Ω is called a conformal factor. The introduction of a conformal factor does not change the causality of M since a curve is timelike (null) for g if and only if it is timelike (null) for $\Omega^2 g$. On