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Taylor's Theorem for Analytic Functions of Operators

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Abstract. We discuss analytic functions on a Banach algebra into itself. In particular expressions for derivatives are given as well as convergent Taylor expansions.

Introduction

The problem of expansion of functions of non-commuting operators occurs in many branches of theoretical physics. Many formal schemes [1–5] have been used, but in very few cases [5] has convergence been established. We discuss a case for which convergence is established. Our approach follows in spirit the work [5] of Araki.

I. Analytic Functions of Operators and Derivatives

Let $F:\mathbb{C}\to\mathbb{C}$ be an analytic function in $G=\{z|\,|z|<\varrho\}$. In the domain G, F has a convergent power series expansion

$$F(z)=\sum_{n=0}^{\infty} c_n z^n . \tag{1}$$

The n^{th} derivative D^nF of F also has a convergent power series having the same domain of convergence as F.

Let $\mathscr{B}$ be a Banach algebra and denote by $\mathscr{L}=\mathscr{L}^1(\mathscr{B})$ the Banach algebra of bounded linear maps L of $\mathscr{B}$ into itself. The norm of $\mathscr{L}^1(\mathscr{B})$ is defined by $\|L\|=\sup\limits_{A\in\mathscr{B}}\dfrac{\|LA\|}{\|A\|}$. We then define the Banach algebras $\mathscr{L}^n(\mathscr{B})$ iteratively by $\mathscr{L}^1(\mathscr{L}^{n-1}(\mathscr{B}))$.

Definition 1. Let $\mathscr{B}$ be a Banach algebra and $A, B\in\mathscr{B}$. For $0\leqq\lambda\leqq 1$ let A_λ be the linear map from $\mathscr{B}$ into $\mathscr{B}$ defined by

$$A_\lambda B=AB-\lambda d_A B , \tag{2}$$