Commun. math. Phys. 47, 221—227 (1976)

Communications in Mathematical Physics

On the Time-Delay of Simple Scattering Systems

Ph. A. Martin

Laboratoire de Physique Théorique, Ecole Polytechnique Fédérale, CH-1006 Lausanne, Switzerland

Abstract. A new rigorous and simple study of the time-delay formula is presented.

I. Introduction

The time-delay of a scattering process may intuitively be considered to be the difference between the time spent by the colliding particles within the region of mutual interaction and the time that they would have spent in the same region had they moved freely. Consider a simple scattering system (H, H_0) on a Hilbert space $\mathscr{H}$ with free evolution $U_t = \exp(-iH_0t)$ and total evolution $V_t = \exp(-iHt)$ [1]. The wave operators $\Omega_\pm = \underset{t\to\pm\infty}{s\text{-lim}} \exp(iHt)\exp(-iH_0t)$ are assumed to be complete and the scattering operator $S = \Omega_+^*\Omega_-$ is unitary. Let P_r be the projection operator on a region Σ_r in configuration space in which the distance between particles does not exceed r. If $\psi_t = V_t\Omega_-\varphi$ is a scattering state which behaves as the freely evolving state $\varphi_t = U_t\varphi$ as $t\to-\infty$, the mean times spent in Σ_r by the interacting and the free particles are respectively:

$$\int_{-\infty}^{\infty} (\psi_t, P_r\psi_t)dt \quad \text{and} \quad \int_{-\infty}^{\infty} (\varphi_t, P_r\varphi_t)dt\,.$$

The time-delay for the scattering state $\Omega_-\varphi$ and the region Σ_r is then defined by [2, 3]:

$$T_r(\varphi) = \int_{-\infty}^{\infty} [(\psi_t, P_r\psi_t) - (\varphi_t, P_r\varphi_t)]dt\,. \tag{1}$$

This definition raises three mathematical questions which are in logical order:

(i) Is the expression (1) meaningful for finite r and suitable states φ?

(ii) Does there exist a time-delay $T(\varphi) = \lim_{r\to\infty} T_r(\varphi)$ for infinite space region?

(iii) Is $T(\varphi)$ related to the classical formula of Eisenbud and Wigner [4] asserting that the time-delay is expressed by the derivative of the phase shift with respect to energy?