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Fredholm Determinants and Inverse Scattering Problems*

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Abstract. The Gel'fand-Levitan and Marchenko formalisms for solving the inverse scattering problem are applied together to a single set of scattering phase-shifts. The result is an identity relating two different types of Fredholm determinant. As an application of the method, an asymptotic formula of high accuracy is derived for a particular Fredholm determinant that determines the level-spacing distribution-function in the theory of random matrices.

I. Statement of the Problem

The inverse scattering problem was solved in the early 1950's by two different methods, one due to Gel'fand and Levitan [1] and the other to Marchenko [2]. The essential difference between the two methods is that Gel'fand and Levitan constructed a scattering potential $V(x)$ on the half-line $0<x<\infty$ by working up from the end $x=0$, while Marchenko worked down from the end $x=\infty$. In the excellent review article of Faddeev [3] and in other discussions of inverse scattering, the methods of Gel'fand-Levitan and Marchenko are presented as alternatives, each by itself being sufficient to construct the potential. Little is said about the consequences of applying both methods to the same scattering data.

The purpose of the present paper is to elucidate the relation between the Gel'fand-Levitan and Marchenko formalisms in the context of a special example. We apply the two methods to a particular scattering problem that happened to arise out of the theory of random matrices [4]. The result of applying the two methods simultaneously is an identity (4.24) linking two Fredholm determinants, one defined on the interval $[0, s]$ and the other on the interval $[s, \infty]$. The determinant on $[0, s]$ is the one that arises naturally in random-matrix theory. The determinant on $[s, \infty]$ is easily expanded into an asymptotic series in negative powers of s. The two inverse-scattering methods together enable us to determine the asymptotic behavior for large s of the random-matrix determinant, with a

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