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On the Uniqueness of the Newtonian Theory as a Geometric Theory of Gravitation

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Abstract. A study is made of geometric theories of gravitation that are consistent with the local validity of Newtonian dynamics. This involves an analysis of the representations of the Galilean group provided by the curvature tensor of a Newtonian spacetime, and by the contravariant mass-momentum tensor. Subject to certain assumptions that are made also in the foundations of general relativity, it is shown that there exists essentially only one such theory that does not place unacceptable restrictions on the mass density of the source. This is the Newtonian theory, generalized by a cosmological term. Any other theory is weaker and is given by a subset of the geometrical equations of the Newtonian theory.

1. Introduction

The spacetime of Newtonian physics in the absence of gravitation may be described covariantly by a symmetric tensor $g^{\alpha\beta}$, a vector t_α and a symmetric affine connexion $\Gamma^\gamma_{\alpha\beta}$. These satisfy

$$g^{\alpha\beta}t_\beta=0\,, \tag{1.1}$$

$$\nabla_\alpha g^{\beta\gamma}=0 \quad \text{and} \quad \nabla_\alpha t_\beta=0\,, \tag{1.2}$$

and

$$R_{\alpha\beta\gamma}{}^{\delta}=0\,, \tag{1.3}$$

where $R_{\alpha\beta\gamma}{}^{\delta}$ and ∇_α are respectively the curvature tensor and covariant derivative operator of the connexion. In addition, $g^{\alpha\beta}$ is required to be positive semi-definite and of matrix rank 3. It follows from these conditions that there exists a family of coordinate systems in which

$$g^{\alpha\beta}=\mathrm{diag}(1,1,1,0) \quad \text{and} \quad t_\alpha=(0,0,0,1)\,, \tag{1.4}$$

and

$$\Gamma^\alpha_{\beta\gamma}=0\,. \tag{1.5}$$

These are related to one another by Galilean transformations, and the connexion with physics is made by identifying them with inertial frames of dynamics.