Commun. math. Phys. 44, 129—132 (1975)

Uncertainty Relations for Information Entropy in Wave Mechanics

Iwo Białynicki-Birula*

Institute of Theoretical Physics, Warsaw University, 00-681 Warsaw, Poland, and Department of Physics, University of Pittsburgh, Pittsburgh, USA

Jerzy Mycielski

Institute of Theoretical Physics, Warsaw University, 00-681 Warsaw, Poland

Received May 10, 1975

Abstract. New uncertainty relations in quantum mechanics are derived. They express restrictions imposed by quantum theory on probability distributions of canonically conjugate variables in terms of corresponding information entropies. The Heisenberg uncertainty relation follows from those inequalities and so does the Gross-Nelson inequality.

The purpose of this paper is to derive a new, stronger version of the Heisenberg uncertainty relation in wave mechanics. This new uncertainty relation has a simple interpretation in terms of information theory. It is also closely related to newly discovered logarithmic Sobolev inequalities.

The new uncertainty relation has the form (for wave functions normalized to unity)

$$-\langle \ln \varrho \rangle - \langle \ln \tilde{\varrho} \rangle \geqq n(1+\ln \pi), \tag{1}$$

where ϱ and $\tilde{\varrho}$ are probability densities in n-dimensional position space and momentum space (or more precisely in wave-vector space),

$$\varrho(\boldsymbol{r}) = |\Psi(\boldsymbol{r})|^2,$$
$$\tilde{\varrho}(\boldsymbol{k}) = |\tilde{\Psi}(\boldsymbol{k})|^2.$$

Brackets $\langle\ \rangle$ denote integration over the whole position space or momentum space with ϱ or $\tilde{\varrho}$. For example

$$\langle \ln \varrho \rangle = \int d^n r \varrho(\boldsymbol{r}) \ln \varrho(\boldsymbol{r}).$$

Our normalization of the Fourier transform is

$$\tilde{\Psi}(\boldsymbol{k}) = \frac{1}{(2\pi)^{n/2}} \int d^n r \exp(-i\boldsymbol{k}\cdot\boldsymbol{r})\Psi(\boldsymbol{r}).$$

It is worth observing that the inequality (1) does not depend on the unit of length used in measuring ϱ and $\tilde{\varrho}$.

Unlike the standard uncertainty relation, which expresses indeterminacy of positions and momenta in terms of the second moments of the corresponding

* Research partly supported by the NSF Grant 36217.