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Invariant States and Conditional Expectations of the Anticommutation Relations

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Abstract. The group G of unitary elements of a maximal abelian von Neumann algebra on a separable, complex Hilbert space H acts as a group of automorphisms on the CAR algebra $\mathscr{A}(H)$ over H. It is shown that the set of G-invariant states is a simplex, isomorphic to the set of regular probability measures on a w^*-compact set S of G-invariant generalized free states. The GNS Hilbert space induced by an arbitrary G-invariant state on $\mathscr{A}(H)$ supports a *-representation of $C(S)$; the canonical map of $\mathscr{A}(H)$ into $C(S)$ can then be locally implemented by a normal, G-invariant conditional expectation.

In this paper we shall define observable Fermion number densities on the spectra of complete one particle observables and study the classical fields which they generate.

Let $\mathscr{A}(H)$ denote the C^*-algebra of the Canonical Anticommutation Relations (CAR) over a complex, separable Hilbert space H. H will be fixed throughout and $\mathscr{A}(H)$ denoted by $\mathscr{A}$. $\mathscr{A}$ is generated algebraically by the range of an antilinear map $f \to a(f)$ of H into $\mathscr{A}$ obeying the CAR:

$$a(f)a(g)+a(g)a(f)=0 \qquad a^*(f)a(g)+a(g)a^*(f)=(g,f) \qquad \forall f,g \in H\,.$$

Let u be a unitary operator on H. Then the map $a(f) \to a(uf)$ extends uniquely to a *-automorphism α_u of $\mathscr{A}$. α_u is called the Bogoliubov automorphism induced by u.

Let $\mathcal{O}$ be a self-adjoint operator on $\mathscr{H}$. $\mathcal{O}$ shall be called complete if its spectral family generates a maximal abelian von Neumann algebra $\mathscr{Y}$ on H. Let (X, B, μ) denote the spectral measure space of $\mathcal{O}$. By the well known isomorphism theorem (I § 7 and III § 1, Corollary 3 of Ref. [3]), completeness of $\mathcal{O}$ leads to identification of $\mathscr{H}$ with $\mathscr{L}^2(X, B, \mu)$ and of $\mathscr{Y}$ with $\mathscr{L}^\infty(X, B, \mu)$.

When $\mathcal{O}$ has discrete spectrum, the number density N on X is defined for each $x \in X$ by $N_x = a^*(\delta_x)a(\delta_x)$ where δ_x is the Kroenecker δ-function at $x \in X$. The number density N generates a classical field which is isomorphic to the lattice gas. One can also isolate the field and density by symmetry considerations (as we have remarked before [16]).

An observable in $\mathscr{A}$ is called $\mathcal{O}$-diagonal if it is diagonal in the Fock representation with respect to the basis formed by anti-symmetric products of eigenvectors

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