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On Commutative Normal $*$-Derivations*

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Abstract. The notion of commutative derivations is introduced into the theory of unbounded derivations in operator algebras. A useful result for the phase transition theory will be shown for these derivations.

§ 1. Introduction

In the papers [1, 2], normal $*$-derivations in uniformly hyperfinite C^*-algebras have been studied. One of the main purposes of those studies is to develop the phase transition theory in general settings and is hopefully to contribute to unsolved problems in the theory.

As the first step to this goal, it is important to study commutative normal $*$-derivations, since they include the Ising models. In the present paper, we shall show a useful result for commutative normal $*$-derivations. We shall discuss applications of this result to the phase transition theory in later papers.

§ 2. A Theorem

Let $\mathfrak{A}$ be a uniformly hyperfinite C^*-algebra and let δ be a normal $*$-derivation in $\mathfrak{A}$ – i.e., there is an increasing sequence $\{\mathfrak{A}_n | n = 1, 2, \ldots\}$ of finite type I-subfactors in $\mathfrak{A}$ such that $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is dense in $\mathfrak{A}$ and the domain $\mathscr{D}(\delta)$ of $\delta = \bigcup_{n=1}^{\infty} \mathfrak{A}_n$. Then there is a sequence (h_n) of self-adjoint elements in $\mathfrak{A}$ such that $\delta(a) = i[h_n, a]$ $(a \in \mathfrak{A}_n)$ for $n = 1, 2, 3, \ldots$.

Definition. A normal $*$-derivation δ in $\mathfrak{A}$ is said to be commutative if we can choose (h_n) as a mutually commuting family. Then we shall show

Theorem. *Suppose that δ is a commutative normal $*$-derivation in $\mathfrak{A}$ and (h_n) is a corresponding family of self-adjoint elements in $\mathfrak{A}$ to δ such that (h_n) is mutually commutative.*

Then there exists a strongly continuous one-parameter subgroup $\{\varrho(t) | -\infty < t < +\infty\}$ of $*$-automorphisms on $\mathfrak{A}$ such that $\varrho(t)(a) = \exp t\delta_{ih_n}(a)$ for $a \in \mathfrak{A}_n (n = 1, 2, 3, \ldots)$ and $-\infty < t < +\infty$, where $\delta_{ih_n}(x) = [ih_n, x]$ $(x \in \mathfrak{A})$. Moreover let δ_1 be the infinitesimal generator of $\{\varrho(t) | -\infty < t < +\infty\}$; then $\delta_1 = \delta$ on $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$.

Proof. Since $h_n - h_{n_0} \in \mathfrak{A}'_{n_0}$ $(n \geqq n_0)$, where $\mathfrak{A}'_{n_0}$ is the commutant of $\mathfrak{A}_{n_0}$ and since $h_n - h_{n_0}$ commutes with h_{n_0}, $h_n - h_{n_0}$ commutes with $\delta^m_{ih_{n_0}}(\mathfrak{A}_{n_0})$ $(m = 1, 2, 3, \ldots)$.

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