Commun. math. Phys. 42, 231—252 (1975)

On the Algebra of Field Operators. The Weak Commutant and Integral Decompositions of States

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Received July 15, 1974; in revised form January 29, 1975

Abstract. It is shown that every continuous representation of a nuclear and separable $*$-algebra on a separable Hilbert space has a certain integral decomposition into representations with a trivial weak commutant. This result is used to obtain a decomposition of Wightman functionals into extremal states.

An example is given of an extremal Wightman functional which does not have the cluster property.

1. Introduction

This paper deals with integral decompositions of representations and states on the algebra of test functions for quantum fields [1–3]. In comparison with the reduction theory for C^*- and v. Neumann algebras one has to face some new problems. To begin with, the positive cone in the dual space of the algebra in question does not have a compact base, so the usual integral representation theory on compact convex sets [4] does not apply. Although it is possible to use the theory of conical measures [4, 5] it is not at all clear whether a maximal conical measure is concentrated on the extremal rays in the dual cone in some sense. Therefore, it seems more promising for a decomposition of a state into extremal states to look at the corresponding representation and decompose with respect to an abelian algebra in the commutant. This is in principle the method we shall use, but there is, however, an essential complication. The operators we have to deal with are in general unbounded, so one has to distinguish between two notions of commutativity, strong and weak. For the decomposition into extremal states, the weak commutant is relevant, because a state is extremal if and only if it defines a representation with a trivial weak commutant. However, unlike the strong commutant, the weak commutant is in general not an algebra, and a decomposition with respect to a maximal abelian algebra in the weak commutant will mostly not result in a decomposition into extremal states. A simple example where this problem occurs is the case of one Hermitean operator with a nonsymmetrical defect index. The solution is well known (see e.g. [6], Appendix I). One considers a self adjoint extension of the operator in a larger Hilbert space and decomposes this extended operator. Our program is to do a similar thing for families of unbounded operators.

Whereas the existence of a suitable extension for one operator is a rather simple matter [6], the main problem we have to solve is to construct such an