Commun. math. Phys. 42, 29—30 (1975)

On the Perturbation of Gibbs Semigroups

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Received July 14, 1974

Abstract. The trace-norm convergence of the Hille-Phillips perturbation series is proved for the whole perturbation class of the generator of a Gibbs semigroup.

In Ref. [1], Uhlenbrock proposed the following terminology:

Definition. A selfadjoint semigroup $\{T(t)\}_{t \geqq 0}$ in a separable Hilbert space with the property:

$$\operatorname{tr} T(t) < \infty, \qquad \forall t > 0 \tag{1}$$

is called a Gibbs semigroup; and raised the problem of proving the trace-norm convergence of the Hille-Phillips perturbation series [2] for a conveniently large class of perturbations of the generator of a Gibbs semigroup. He gave also a proof of trace-norm convergence in the case of bounded perturbations, based on an inequality due to Ginibre and Gruber [3].

The aim of this note is to point out that a slight modification of this very argument allows to prove the trace-norm convergence of the series for the whole Hille-Phillips perturbation class.

Proposition. *Let $T(t)$ be a Gibbs semigroup and A its generator. Let B be A-bounded and such that:*

$$\int_0^1 \|B\,T(t)\|\,dt < \infty\,. \tag{2}$$

Then the series:

$$S(t) = \sum_{n=0}^{\infty} S_n(t) \tag{3}$$

with:

$$S_0(t) = T(t)\,; \qquad S_n(t) = \int_0^t ds\, S_0(t-s)\, B\, S_{n-1}(s) \tag{4}$$

is $\|\cdot\|_1$-convergent uniformly for t in compact subsets of $(0, \infty)$. In particular, if B is moreover symmetric, then $S(t)$ is a Gibbs semigroup.

Proof. If B is A-bounded, then $B\,T(t) = [B\,R(\lambda, A)]\,[(\lambda - A)\,T(t)]$ is bounded and condition (2) makes sense. One can write $S_n(t)$ as a multiple (trace-norm) Bôchner integral:

$$S_n(t) = \int \cdots \int ds_1 \ldots ds_n \chi_n^t(s_0, s_1, \ldots, s_n)\, S_0(s_0)\, B\, S_0(s_1) \ldots B\, S_0(s_n)\,, \tag{5}$$