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Wigner's Theorem on Symmetries in Indefinite Metric Spaces*

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Abstract. The description of symmetries in indefinite metric spaces is investigated. It is shown that ray transformations preserving the modulus of an indefinite scalar product can be implemented by linear or antilinear vector transformations which are generalized unitary or antiunitary operators with respect to the indefinite scalar product. A number of interesting features arise since such operators need not to be bounded.

1. Introduction

Wigner's theorem [1, 2] on ray transformations in Hilbert spaces plays a fundamental rôle in the foundations of quantum mechanics [3, 4] and it has deep connections with the mathematical theory of projective spaces [5–7]. In the proof of the theorem a crucial rôle is played by the Hilbert space structure and, in particular, by the positivity of the scalar product. On the other hand it has become more and more evident that indefinite metric spaces may be more useful both for the discussion of physical problems as well as for more genuine mathematical questions. As far as the physical applications are concerned the growing evidence comes mainly from the theory of quantum fields for which the use of an indefinite metric has been advocated several times in the past [8–10] as a solution of the divergence problem in quantum field theory [11, 12] and as a method to obtain better regularity properties for theories previously regarded as untractable [13–16]. For several physically interesting theories the use of an indefinite metric is not only a promising suggestion but an unavoidable feature if one wants to preserve some basic properties of the fields like relativistic covariance and locality [17–19].

From a mathematical point of view the interest of indefinite metric spaces has been pioneered by the Russian mathematicians [20–22] and we refer to their papers for general motivations as well as for an exposition of the results obtained in that field. It may be interesting to stress that indefinite metric spaces appear very useful also in solving stability problems in the classical theory of damped oscillations and in general as a powerful tool for solving systems of differential equations [24]. In particular canonical linear differential equations with a periodic Hamiltonian have been studied with success using indefinite metric spaces [25].

In the following by an indefinite metric space we mean an Hilbert space H equipped with a bounded symmetric sesquilinear form $\langle\cdot,\cdot\rangle=(\cdot,\eta\cdot)$, where $(\cdot,\cdot)$

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