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Global Conformal Invariance in Quantum Field Theory

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Abstract. Suppose that there is given a Wightman quantum field theory (QFT) whose Euclidean Green functions are invariant under the Euclidean conformal group $\mathfrak{G} \simeq SO_e(5, 1)$. We show that its Hilbert space of physical states carries then a unitary representation of the universal (∞-sheeted) covering group $\mathfrak{G}^*$ of the Minkowskian conformal group $SO_e(4, 2)/\mathbb{Z}_2$. The Wightman functions can be analytically continued to a domain of holomorphy which has as a real boundary an ∞-sheeted covering $\tilde{M}$ of Minkowski-space M^4. It is known that $\mathfrak{G}^*$ can act on this space $\tilde{M}$ and that $\tilde{M}$ admits a globally $\mathfrak{G}^*$-invariant causal ordering; $\tilde{M}$ is thus the natural space on which a globally $\mathfrak{G}^*$- invariant local QFT could live. We discuss some of the properties of such a theory, in particular the spectrum of the conformal Hamiltonian $H = \frac{1}{2}(P^0 + K^0)$.

As a tool we use a generalized Hille-Yosida theorem for Lie semigroups. Such a theorem is stated and proven in Appendix C. It enables us to analytically continue contractive representations of a certain maximal subsemigroup $\mathfrak{S}$ of $\mathfrak{G}$ to unitary representations of $\mathfrak{G}^*$.

1. Introduction

Conformal invariant quantum field theory (QFT) is of interest from the point of view of constructive quantum field theory because such theories can be analyzed to a remarkable extent by nonperturbative methods, i.e. without recourse to iterative techniques [1, 2]. One adopts the usual postulates of local QFT (Wightman axioms [3]): Spectrum condition, positivity, and locality. In addition one demands that the Euclidean Green functions are invariant under the Euclidean conformal group[1] $SO_e(5, 1)$. [The Euclidean Green functions are obtained by analytically continuing the vacuum expectation values of fields. (Wightman functions) to imaginary times.] This is the hypothesis of "weak conformal invariance" [4]. It implies that the Wightman functions are invariant under infinitesimal conformal transformations in Minkowski space.

The hypothesis of weak conformal invariance was invented because of the familiar difficulties with global conformal transformations in Minkowski space M^4. Such a transformation can take points to infinity, and it does not help either to compactify Minkowski space, i.e. add points at infinity, because the resulting manifold M_c^4 contains closed timelike curves and so does not admit a global causal ordering [5].

In the present paper we consider Wightman field theories with a unique vacuum, with Wightman functions that are (tempered) distributions, and whose Euclidean Green functions are conformal invariant as explained above. We show that the Hilbert space of physical states of any such theory carries a unitary representation of the ∞-sheeted universal covering group $\mathfrak{G}^*$ of the Minkowskian

[1] The subscript e identifies the identity component of the group.