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Completely Positive Maps and Entropy Inequalities

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Abstract. It is proved that the relative entropy for a quantum system is non-increasing under a trace-preserving completely positive map. The proof is based on the strong subadditivity property of the quantum-mechanical entropy.

The object of this note is to prove that the relative entropy functional for a finite quantum system is nonincreasing under a trace-preserving completely positive map of the state space into itself. This theorem is a generalization of an earlier result for expectations [1] (since expectations are completely positive maps [2]) which is in its turn a generalization of a well-known theorem in information theory [3, 4]. The proof is based on the strong subadditivity property of the quantum-mechanical entropy which was derived recently by Lieb and Ruskai [5] from certain trace inequalities proved by Lieb [6] and, in an alternative way, by Epstein [7].

The physical interest of completely positive maps lies in the theory of measurements and the operational approach to quantum mechanics [8, 9]. We will give some simple arguments that the operations should be chosen to be completely positive.

Denote by $B(\mathscr{H})$ the bounded operators in a separable Hilbert space $\mathscr{H}$, by $T(\mathscr{H})$ the trace class operators in $\mathscr{H}$ and by $T_+(\mathscr{H})$ the positive elements in $T(\mathscr{H})$. Furthermore, let $\mathscr{M}_n$ be the algebra of $n \times n$ complex matrices.

Let $A, B \in T_+(\mathscr{H})$. Define the operator-valued entropy by $\hat{S}(A) = -A \ln A$.

Let $\lambda \in (0, 1)$ and define

$$\hat{S}_\lambda(A|B) = \lambda^{-1} [\hat{S}(\lambda A + (1-\lambda)B) - \lambda \hat{S}(A) - (1-\lambda)\hat{S}(B)]$$

$$S_\lambda(A|B) = \operatorname{Tr} \hat{S}_\lambda(A|B).$$

The relative entropy is defined by

$$S(A|B) = \lim_{\lambda \to 0} S_\lambda(A|B).$$

From Lemma 4 of [10] it follows that this definition is equivalent to that used in [1, 10].

We know that $\hat{S}_\lambda(A|B)$ is positive [10], hence the trace is well-defined, eventually infinite. When $\lambda \downarrow 0$, $\hat{S}_\lambda(A|B)$ is monotonously increasing,