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On Infinite Direct Products of Continuous Unitary One-Parameter Groups

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Abstract. We discuss the infinite product of unitary operators in an incomplete direct product of Hilbert spaces. Necessary and sufficient conditions are derived under which this infinite product leads to a continuous unitary one-parameter group provided each factor is assumed to have this property. A certain minimal condition guarantees the existence of a renormalized unitary group. An application is made to product representations of the canonical commutation relations in order to determine the admissible test functions.

1. Introduction

Our investigations about infinite products of unitary one-parameter groups are motivated by the following physical situation. Consider a system which consists of infinitely many dynamically independent subsystems. The time evolution of the subsystems may be described by unitary operators $U_r(t)$ acting in separable Hilbert spaces H_r; $r = 1, 2, \ldots$ labelling the subsystems. Independence of the various subsystems is achieved most simply if one takes as representation space H for the total system an incomplete direct product of the Hilbert spaces H_r, $H = \bigotimes_r (H_r, \phi_r)$ [1]. $\{\phi_r\}_{r=1}^{\infty}$, a sequence of unit vectors with $\phi_r \in H_r$, which is called reference vector determines a separable subspace of the nonseparable complete direct product. The (naively suggested) time evolution of the total system should then be given by $U(t) = \bigotimes_r U_r(t)$. However, in general the infinite product of continuous one-parameter groups of unitary operators does not lead to a continuous unitary one-parameter group. $U(t) = \bigotimes_r U_r(t)$ may happen to be not even unitary in $H = \bigotimes_r (H_r, \phi_r)$.

So, we are dealing with a family $\{U_r(\lambda)\}_{r=1}^{\infty}$ of unitary operators $U_r(\lambda)$, $\lambda \in \mathbb{R}$, acting in separable Hilbert spaces H_r such that

$$U_r(\lambda)\, U_r(\lambda') = U_r(\lambda + \lambda'), \tag{1.1}$$