Commun. math. Phys. 39, 111—119 (1974)

Expectations and Entropy Inequalities for Finite Quantum Systems

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Received March 19, 1974; in revised form July 16, 1974

Abstract. We prove that the relative entropy is decreasing under a trace-preserving expectation in $B(\mathscr{K}^1)$, and we show the connection between this theorem and the strong subadditivity of the entropy. It is also proved that a linear, positive, trace-preserving map Φ of $B(\mathscr{K})$ into itself such that $\|\Phi\| \leqq 1$ decreases the value of any convex trace function.

The main object of this note is to prove that the relative entropy is decreasing under a trace-preserving expectation from $B(\mathscr{K})$ to a von Neumann subalgebra (Theorem 1). We will show the connection between this theorem and the property of strong subadditivity of the entropy functional in quantum statistical mechanics [1]. The theorem is a generalization of a result by Umegaki [2] [for the case $B(\mathscr{H})$] and hence of an inequality in information theory [3]. The proof rests on a result by Lieb [4] on a generalized Wigner- Yanase- Dyson inequality.

The intuitive content of Theorem 1 is that an expectation always decreases the information content of the states, especially it makes it more difficult to distinguish two states from each other. Theorem 2 makes a similar but weaker statement for a larger class of maps: a positive, tracepreserving map of $B(\mathscr{K})$ into itself with norm at most equal to one decreases the value of any convex trace function on $B(\mathscr{K})$.

Let $A, B \in T_+(\mathscr{K})$ (the positive trace class operators in a separable Hilbert space $\mathscr{K}$). The *entropy* of A is defined by

$$S(A) = \operatorname{Tr} \hat{S}(A), \qquad \hat{S}(A) = -A \log A .$$

If $\{|i\rangle\}$ is a complete orthonormal set of eigenvectors of A or B then we can define the *relative entropy*[2] through

$$S(A \mid B) = \Sigma \langle i| \, (A \log A - A \log B + B - A) \, |i\rangle$$

(see [5] for details). In [5] it was shown that if $S(A \mid B) < \infty$ we have

$$S(A \mid B) = \operatorname{Tr} \hat{S}(A \mid B)$$

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[1] For $\mathscr{K}$ read $\mathscr{H}$ throughout.

[2] In [5] this was called the *conditional entropy*.