Commun. math. Phys. 37, 175—182 (1974)

A Generalised Entropy Function

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Received December 6, 1973

Abstract. Let φ be a faithful normal semi-finite weight on a von Neumann algebra $\mathscr{M}$. Normal states on $\mathscr{M}$ almost majorised by this weight are defined. For this class of states on $\mathscr{M}$ a theorem is proved. Using this result we define entropy of normal states on $\mathscr{M}$ and we show that this entropy function generalises the entropy both of classical and of quantum statistical mechanics.

1. Introduction

Let φ and ψ be two normal states on a von Neumann algebra $\mathscr{M}$. Suppose ψ is faithful. In Ref. [1] Dixmier introduces the notion of the state φ being almost majorised by the state ψ. He remarks that to any state φ almost majorised by ψ corresponds a closable operator affiliated with $\pi_\psi(\mathscr{M})'$, where π_ψ is the *-representation of $\mathscr{M}$ associated with ψ by the G.N.S.-construction.

We define when a normal state φ on $\mathscr{M}$ is almost majorised by a faithful normal semi-finite weight ψ on $\mathscr{M}$.

Using some results of Perdrizet [2], we show that with any state φ almost majorised by the weight ψ can be associated in a unique way a positive self-adjoint operator affiliated with $\pi_\psi(\mathscr{M})'$.

This result is used to define a generalised entropy function. The phase space of a system in classical statistical mechanics is a measure space M, ν. The measure ν gives the a priori probability of the points of M. The macroscopic states of the system are described by positive normalised measures μ on M which are absolutely continuous with respect to the measure ν. To each such measure μ corresponds a positive integrable function f on M which satisfies $\int f\, d\nu = 1$ and $d\mu = f\, d\nu$. These functions f are called density functions and the entropy of the measure μ is given by the expression

$$S(\mu) = -\int f \log f\, d\nu .$$

Let $\mathscr{H}$ be the Hilbert space of wave functions of a quantum mechanical system. In many cases the statistical states of the system are described by the normal states on the space $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$. To each normal state ψ on $\mathscr{B}(\mathscr{H})$ corresponds a unique density