Commun. math. Phys. 36, 287—304 (1974)

Product States for Local Algebras

Detlev Buchholz
II. Institut für Theoretische Physik der Universität Hamburg,
Hamburg, Federal Republic of Germany

Received February 8, 1974

Abstract. Let $\mathscr{R}(\mathcal{O}_1) \subset \mathscr{B}(\mathscr{H})$ and $\mathscr{R}(\mathcal{O}_2) \subset \mathscr{B}(\mathscr{H})$ be the von Neumann algebras associated with the space-time regions $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively in the vacuum representation of the free neutral massive scalar field. For suitably chosen spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ it is proved that there exists a normal product state φ of $\mathscr{B}(\mathscr{H})$,

$$\varphi(AB) = \varphi(A) \cdot \varphi(B) \quad \text{for all} \quad A \in \mathscr{R}(\mathcal{O}_1) \quad \text{and} \quad B \in \mathscr{R}(\mathcal{O}_2).$$

Some consequences for the algebraic structure of the local rings are pointed out.

I. Introduction

It has been shown by Roos [1] that the local C^*-algebras $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ associated with spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ in Minkowski space are statistically independent: every pair of states φ_1 of $\mathfrak{A}(\mathcal{O}_1)$ and φ_2 of $\mathfrak{A}(\mathcal{O}_2)$ can be extended to a state φ of $\mathfrak{A}$ (the C^*-algebra generated by all local C^*-algebras). Moreover, φ can be chosen to be a product state for $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$:

$$\varphi(AB) = \varphi(A) \cdot \varphi(B) \quad \text{for all} \quad A \in \mathfrak{A}(\mathcal{O}_1) \quad \text{and} \quad B \in \mathfrak{A}(\mathcal{O}_2).$$

For a general structure analysis of physically interesting representations of the local C^*-algebras it would be important to know whether one can extend the result of Roos in the following way: let $\mathscr{R}(\mathcal{O}_1)$ and $\mathscr{R}(\mathcal{O}_2)$ be von Neumann sub-algebras of $\mathscr{B}(\mathscr{H})$ [1] associated with the regions $\mathcal{O}_1$ and $\mathcal{O}_2$. Take any pair of normal states φ_1 of $\mathscr{R}(\mathcal{O}_1)$ and φ_2 of $\mathscr{R}(\mathcal{O}_2)$; does there exist a normal state φ of $\mathscr{B}(\mathscr{H})$ which is an extension of φ_1 and φ_2 and a product state for $\mathscr{R}(\mathcal{O}_1)$ and $\mathscr{R}(\mathcal{O}_2)$? If this question had a positive answer the local rings would have the following remarkable properties:

a) From the existence of normal product states one could conclude that $\mathscr{R}(\mathscr{R}(\mathcal{O}_1), \mathscr{R}(\mathcal{O}_2))$ (the von Neumann algebra generated by $\mathscr{R}(\mathcal{O}_1)$ and $\mathscr{R}(\mathcal{O}_2)$) is isomorphic to the W^*-tensor product of $\mathscr{R}(\mathcal{O}_1)$ and $\mathscr{R}(\mathcal{O}_2)$. Locality would reflect itself in a very simple algebraic structure of $\mathscr{R}(\mathscr{R}(\mathcal{O}_1), \mathscr{R}(\mathcal{O}_2))$.

[1] $\mathscr{B}(\mathscr{H})$ is the algebra of all bounded operators in the representation space $\mathscr{H}$ of $\mathfrak{A}$.