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Remarks on the FKG Inequalities

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Abstract. The FKG inequalities are generalized to two probability distributions. A theorem is proved which shows how one distribution dominates the other and makes it clear why expectation values of increasing functions with respect to one distribution are larger than with respect to the other.

Let Γ be a finite distributive lattice and let μ_1 and μ_2 be probability distributions on Γ. One of the most common applications of the FKG inequalities [3] is to find conditions which guarantee that for all functions f on Γ such that $x \leqq y$ implies $f(x) \leqq f(y)$ one has

$$\sum_{x \in \Gamma} f(x)\,\mu_1(x) \geqq \sum_{x \in \Gamma} f(x)\,\mu_2(x). \tag{1}$$

The inequality (1) seems to be saying that somehow the distribution μ_1 is situated higher up on the lattice than μ_2 is. We prove here a theorem making this precise. The theorem is also strong enough to imply inequalities such as (1) and the FKG inequalities [Corollary (12) below].

(2) **Lemma.** *Let Γ be a finite distributive lattice. Then Γ is isomorphic to a sublattice, $\tilde{\Gamma}$, of the lattice of subsets of a finite set Λ. Moreover, Λ and $\tilde{\Gamma}$ can be chosen in such a way that $\emptyset$ and Λ are in $\tilde{\Gamma}$ and for all $A, B \in \tilde{\Gamma}$ there is a sequence $A = A_0, A_1, \ldots, A_n = B$ in $\tilde{\Gamma}$ such that*

$$|A_i \triangle A_{i+1}| = 1 \quad \textit{for all } i.$$

Here $|A|$ denotes the cardinality of A, and

$$A \triangle B = (A \backslash B) \cup (B \backslash A).$$

Lemma (2) is essentially Corollary (2), Page 59 in [1]. Actually Corollary (2) in [1] is not phrased this way; however, it is easily seen from the proof to be equivalent to Lemma (2).

For the rest of this note Λ will be a fixed finite set and Γ will be a sublattice of the lattice of subsets of Λ. It will also be assumed that Γ has the properties of $\tilde{\Gamma}$ mentioned in Lemma (2).

Let μ be a strictly positive probability distribution on Γ. That is, $\mu(A) > 0$ for all $A \in \Gamma$ and $\sum_{A \in \Gamma} \mu(A) = 1$.