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Support Properties of the Free Measure for Boson Fields

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Abstract. Let μ be the measure on $\mathscr{S}'(\mathbb{R}^d)$ corresponding to the Gaussian process with mean zero and covariance $(f, (-\Delta+1)^{-1} g)$ on $\mathscr{S}(\mathbb{R}^d)$. It is proven that the set

$$(-\Delta_{d-1}+1)^{d/4-\frac{1}{2}+\alpha}(1+x^2)^{d/4}[\log(2+x^2)]^{\beta} L^2(\mathbb{R}^d)$$

has μ measure one if $\alpha>0$ and $\beta>\frac{1}{2}$ and μ measure zero if $\alpha>0$ and $\beta<\frac{1}{2}$; here Δ_{d-1} is the Laplacian in any $d-1$ dimensions when $d>1$ and $\Delta_0=\Delta$.

§ 1. Introduction

Nelson [11] has shown that to every generalized stochastic process ϕ over $\mathscr{S}(\mathbb{R}^d)$ satisfying the axioms of a Euclidean Markov field theory there corresponds in a natural way a relativistic Boson field theory satisfying the Wightman axioms. In particular, the Gaussian process of mean zero with covariance $\langle\phi(f)\,\phi(g)\rangle=(f,(-\Delta+1)^{-1}g)$ corresponds to the free Boson field of mass 1 in d-dimensional space-time [12]. By Minlos' theorem [6, 8] this "free process" can be realized on $\mathscr{S}'(\mathbb{R}^d)$, the topological dual of the nuclear space $\mathscr{S}(\mathbb{R}^d)$. That is, there is a Borel measure μ on $\mathscr{S}'(\mathbb{R}^d)$ so that

$$e^{-\frac{1}{2}(f,(-\Delta+1)^{-1}f)}=\int_{\mathscr{S}'(\mathbb{R}^d)} e^{i\langle q,f\rangle}\,d\mu(q)$$

for all $f\in\mathscr{S}(\mathbb{R}^d)$. In this realization, the free process itself is given by the function $\phi(f): q\to\langle q,f\rangle$ for $q\in\mathscr{S}'$.

The processes corresponding to interacting Boson theories are usually constructed by taking limits of non-Gaussian perturbations of the "free measure" μ (see, for example, [10]). The properties of the free measure play an important role in this procedure [4, 7]. Thus it seems useful to study the free process in order to gain insight into and develop analytic tools for the more complicated interacting processes.

In this paper we investigate the support properties of μ by elementary methods. The point is that $\mathscr{S}'$ is an unnecessarily large space in which to