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Inner Automorphisms of von Neumann Algebras

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Abstract. It is first shown that a *-automorphism of a factor is inner if and only if it is asymptotically equal to the identity automorphism. Then it is shown that a periodic *-automorphism of a von Neumann algebra $\mathscr{R}$ is inner if and only if its fixed point algebra is a normal subalgebra of $\mathscr{R}$.

1. Introduction

It is often of importance to know whether a *-automorphism of a von Neumann algebra is inner or not. In the present paper we shall study two aspects of this problem. The first results essentially state that a *-automorphism α of a factor $\mathscr{R}$ is inner if and only if it is asymptotically equal to the identity automorphism ι. By this we mean that if $\varepsilon > 0$ is sufficiently small, then there is a type I subfactor $\mathscr{m}$ of $\mathscr{R}$ such that $\|(\alpha - \iota) | \mathscr{m}^c\| < \varepsilon$, where $\mathscr{m}^c = \mathscr{m}' \cap \mathscr{R}$. A similar theorem has been obtained by Lance [7] for UHF-algebras. The second set of results combine innerness with properties of the fixed point algebra $\mathscr{R}^\alpha$ of α. The main result says that a necessary and sufficient condition for a periodic α to be inner is that $\mathscr{R}^\alpha$ is normal in $\mathscr{R}$, i.e. $\mathscr{R}^\alpha = \mathscr{R}^{\alpha cc}$. The first results are for simplicity stated for factors while the latter are proved for general von Neumann algebras.

2. Asymptotic Properties

In this section we prove the asymptotic theorems mentioned in the introduction. The key result is the following lemma; ι will here and later denote the identity automorphism.

Lemma 2.1. *Let $\mathscr{R}$ be a factor, α a *-automorphism of $\mathscr{R}$, and* $0 < \varepsilon < 1/1800$. *Suppose there is a type I subfactor $\mathscr{m}$ of $\mathscr{R}$ such that* $\|(\alpha - \iota) | \mathscr{m}^c\| < \varepsilon$. *Then α is inner.*

Proof. We first show $\|\mathscr{m} - \alpha(\mathscr{m})\| \leqq 6\varepsilon$, where $\|\mathfrak{A} - \mathscr{B}\|$ denotes the distance between two *-algebras, i.e.

$$\|\mathfrak{A} - \mathscr{B}\| = \sup\{\delta(A, \mathscr{B}_1), \delta(B, \mathfrak{A}_1) : \|A\| \leqq 1, A \in \mathfrak{A}, \|B\| \leqq 1, B \in \mathscr{B}\}$$

where $\delta(A, \mathscr{B}_1) = \inf\{\|A - B\| : B \in \mathscr{B}, \|B\| \leqq 1\}$, see [5].