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The Classical Limit for Quantum Mechanical Correlation Functions

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Abstract. For quantum systems of finitely many particles as well as for boson quantum field theories, the classical limit of the expectation values of products of Weyl operators, translated in time by the quantum mechanical Hamiltonian and taken in coherent states centered in x- and p-space around $\hbar^{-1/2}$ (coordinates of a point in classical phase space) are shown to become the exponentials of coordinate functions of the classical orbit in phase space. In the same sense, $\hbar^{-1/2}$ [(quantum operator) (t) − (classical function) (t)] converges to the solution of the linear quantum mechanical system, which is obtained by linearizing the non-linear Heisenberg equations of motion around the classical orbit.

§ 1. Introduction

Consider the canonical system with the real Hamilton function

$$\mathscr{H}(\pi, \xi) = \pi^2/2m + V(\xi) \tag{1.1}$$

in the $2f$-dimensional phase space $\mathbb{R}^{2f} \ni (\pi, \xi)$. If $\operatorname{grad} V = \nabla V$ is Lipschitz around ξ, then the canonical equations

$$m\dot{\xi}(t) = \pi(t), \qquad \dot{\pi}(t) = -\operatorname{grad} V(\xi(t)) \tag{1.2}$$

have a unique solution $(\xi(\alpha, t), \pi(\alpha, t))$ for times $|t| < T(\alpha)$ (possibly $0 < T(\alpha) \leqq \infty$) with the initial data

$$\xi(\alpha, 0) = \xi, \qquad \pi(\alpha, 0) = \pi, \qquad \alpha = (\xi + i\pi)/\sqrt{2}. \tag{1.3}$$

While the classical equations (1.2) have locally unique but globally possibly nonexistent solutions (escape to infinity in finite times or collisions in the N-body problem), the corresponding quantum mechanical problem

$$i\hbar \frac{\partial \psi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \Delta \psi(x, t) + V(x)\, \psi(x, t) \tag{1.4}$$

in $L^2(\mathbb{R}^f)$ has always global solutions, if $p_\hbar^2/2m$ and $V_\hbar$ have a common dense domain $\mathscr{D}$ and if $\psi = \psi(\cdot, 0) \in \mathscr{D}$, by taking any selfadjoint extension $H_\hbar$ of the real and symmetric operator $p_\hbar^2/2m + V_\hbar$, $U_\hbar(t) = \exp(-iH_\hbar t/\hbar)$