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Continuous Sample Paths in Quantum Field Theory

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Abstract. Nelson's free Markoff field on $\mathbb{R}^{l+1}$ is a natural generalization of the Ornstein-Uhlenbeck process on $\mathbb{R}^1$, mapping a class of distributions $\phi(\boldsymbol{x}, t)$ on $\mathbb{R}^l \times \mathbb{R}^1$ to mean zero Gaussian random variables ϕ with covariance given by the inner product $\left(\left(m^2 - \Delta - \frac{\partial^2}{\partial t^2}\right)^{-1} \cdot, \cdot\right)_2$. The random variables ϕ can be considered functions $\phi\langle q\rangle = \int \phi(\boldsymbol{x}, t)\, q(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt$ on a space of functions $q(\boldsymbol{x}, t)$. In the O.U. case, $l = 0$, the classical Wiener theorem asserts that the underlying measure space can be taken as the space of continuous paths $t \to q(t)$. We find analogues of this, in the cases $l > 0$, which assert that the underlying measure space of the random variables ϕ which have support in a bounded region of $\mathbb{R}^{l+1}$ can be taken as a space of continuous paths $t \to q(\cdot, t)$ taking values in certain Soboleff spaces.

Introduction

The Feynman-Kac formula, which solves the Schrödinger equation by giving its imaginary time Green's function as an integral over Wiener space, has an infinite-number-of-degrees-of-freedom analogue which solves regularized Boson quantum field theories by giving their Schwinger function as integrals over the probability space associated with the free Markoff field (cf. [1–4]). Detailed structure of the associated probability space is not needed to obtain the basic formulae; but Wiener's theorem on the continuity of sample paths is of well known usefulness in the case of a finite number of degrees of freedom. Here we shall attempt to find its analogue in the infinite case.

The free Markoff field was introduced by Nelson [5] who used only an abstract representation of the underlying probability space but indexed the Gaussian random variables by the elements of the Hilbert space $\mathfrak{H}$ of real distributions on $\mathbb{R}^{l+1}$ in the norm $\|(m^2 - \Delta)^{-1/2} \cdot \|_2$, where Δ is the Laplacian on $\mathbb{R}^{l+1}$. That is, each element of the Hilbert space was made a Gaussian random variable of mean zero so that the covariance of two elements would be their inner product. We will take the dual point of view and begin with the underlying "probability space" as the Hilbert space $\mathfrak{Q}$ of real functions in the norm $\|(m^2 - \Delta)^{1/2} \cdot \|_2$