Commun. math. Phys. 35, 151—154 (1974)

The Convergence of BPH Renormalization

Eugene R. Speer
Department of Mathematics, Rutgers University, New Brunswick, USA

Received July 30, 1973

Abstract. The convergence of the integrals defining BPH renormalized Feynman amplitudes is derived from the known additive structure of analytic renormalization.

In this paper we derive the convergence of BPH renormalization [1–3] from the known additive structure of analytic renormalization [4], providing an alternate and perhaps simpler route to this important result. We adopt without further remark the notation of [1, 4].

Suppose that $f(\boldsymbol{\lambda})$ is meromorphic in $\mathbb{C}^L$, with at most simple poles on varieties $\Lambda(\chi)=0, \pm 1, \pm 2, \ldots$, where for $\chi \subset \{1, \ldots L\}$, $\Lambda(\chi) = \sum_{l \in \chi} (\lambda_l - 1)$. For $\boldsymbol{\kappa} \in \mathbb{C}^L$, let $\mathscr{V}^\kappa$ be the analytic evaluator of [4; 3.4 (b)], but defined with center $\boldsymbol{\kappa}$: choosing $0 < R_1 < \cdots < R_L \ll 1$ to satisfy $\sum_{i<j} R_i < R_j$, and defining C_i^j as the contour $|\mu_j - \kappa_j| = R_i$,

$$\mathscr{V}^\kappa f(\boldsymbol{\lambda}) = \frac{(2\pi i)^{-L}}{L!} \sum_{s \in S_L} \int_{C^1_{s(1)}} d\mu_1 \ldots \int_{C^L_{s(L)}} d\mu_L \frac{f(\boldsymbol{\mu})}{(\mu_1-\lambda_1)\ldots(\mu_L-\lambda_L)}$$

whenever $|\lambda_l - \kappa_l| < R_1$. $\mathscr{V}^\kappa f$ is analytic at $\boldsymbol{\kappa}$.

Now let G be a Feynman graph with vertices $V_1, \ldots V_m$ and lines $\{1, \ldots L\}$. If $\hat{\mathscr{X}}$ is a set of vertex parts for G, $U = \{V_1' \ldots V_r'\}$ a generalized vertex, and $Q = \{U_1, \ldots U_s\}$ a partition of U, $\mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1' \ldots V_r')$ is the amplitude defined for $\mathrm{Re}\lambda_l \gg 0$ by $\mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1' \ldots V_r') = \prod_1^s \hat{\mathscr{X}}(U_i) \prod_{\mathrm{conn}} \Delta_l$.

Theorem 1. *If* $\boldsymbol{\kappa} \in C^L$ *satisfies*

$$\mathrm{Re}\,\kappa_l \geqq 1, \qquad l = 1, \ldots L, \tag{1}$$

then

$$\mathscr{V}^\kappa \mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1', \ldots V_r') = \sum_R \mathscr{T}_{R,\tilde{\mathscr{X}}(Q,\hat{\mathscr{X}})}(V_1', \ldots V_r'), \tag{2}$$

where the $\tilde{\mathscr{X}}$*'s are new vertex parts, and the sum is over partitions* R *of* $\{V_1' \ldots V_r'\}$ *at least as coarse as* Q. *Note in particular that if* $Q = \{U\}$, $\tilde{\mathscr{X}}(Q, \hat{\mathscr{X}})(V_1' \ldots V_r') = \mathscr{V}^\kappa \hat{\mathscr{X}}(V_1' \ldots V_r')$.

Proof. As in [4, § 4]. The change of center to $\boldsymbol{\kappa}$ and the extension to a generalized graph introduce only a notational difference in the proof.