Commun. math. Phys. 32, 245—268 (1973)

The Transfer Matrix for a Pure Phase in the Two-dimensional Ising Model

D. B. Abraham

Theoretical Chemistry Department, University of Oxford

A. Martin-Löf

Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden

Received March 10, 1973

Abstract. The problem of diagonalizing the transfer matrix for the two dimensional Ising model with all boundary spins equal to $+1$ is solved by use of the spinor method. This provides a simple proof that the spontaneous magnetization is actually given by the well known formula for the long range order with toroidal boundary conditions, and this means that the critical temperature is precisely that temperature above which the state is unique and below which it is non unique. An expression for the magnetization at finite distance from the boundary is also given, and a simple derivation of the formula for the surface tension between two coexisting phases is presented. Finally the relation between the degeneracy of the spectrum and the phase transition is discussed.

Introduction

In this paper we consider the problem of calculating the pair correlations and the spontaneous magnetization of a two-dimensional Ising spin system with nearest neighbour interaction and no external field in a rectangular box Λ completely surrounded by $+$ spins. The calculation is achieved by the use of the transfer matrix method appropriately modified to account for the boundary conditions chosen.

The significance of considering the situation where all spins on the boundary of the box are fixed to $+1$ depends on the following facts, which can be proved by simple arguments using the G.K.S. inequalities [9, 10]:

In the presence of an external field $h \geqq 0$ all the correlations $\langle \sigma_A \rangle_{h,\Lambda,+} \equiv \left\langle \prod_{p \in A} \sigma_p \right\rangle_{h,\Lambda,+}$ have limits $\langle \sigma_A \rangle_{h,+}$ as the sides of the box tend to infinity, (when $h > 0$ the limits are completely independent of the boundary condition). The $\langle \sigma_A \rangle_{h,+}$ are translationally invariant and

$$\lim_{d(A,B) \to \infty} \langle \sigma_A \sigma_B \rangle_{h,+} = \langle \sigma_A \rangle_{h,+} \langle \sigma_B \rangle_{h,+} \,.$$