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The Spherical Hierarchical Model

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Abstract. The spherical version of Dyson's hierarchical model is analyzed. A particular case which is designed to simulate the long-range Ising problem is dealt with in detail. A phase transition is found with critical temperature

$$\beta_c = \tfrac{1}{2}(2^\alpha - 2)(4 - 2^\alpha)^{-1}$$

where n^{th} neighbor spins interact with a strength of $n^{-\alpha}$. Critical exponents are calculated for this particular case and are found to be identical with the critical exponents of the long-range spherical Ising model.

I. Introduction

The hierarchical model suggested by Dyson [1] is defined as follows: "There are 2^p spins $\mu_j = \pm 1$, labelled by the index $j = 1, 2 \ldots 2^N$. For each pair of integers $p = 0, 1, 2 \ldots N$; $r = 1, 2 \ldots 2^{N-p}$ we consider the spin sum

$$S_{p,r} = \sum_j \mu_j, \qquad (r-1)\,2^p + 1 \leqq j \leqq r\,2^p. \tag{1}$$

This is the sum of the r^{th} block of 2^p consecutive spins. The hierarchical character of these sums is expressed by the relation

$$S_{p,r} = S_{p-1,2r-1} + S_{p-1,2r}, \qquad p = 1, 2 \ldots N. \tag{2}$$

We assume the interaction energy in the model to be

$$H_N = -\sum_{p=1}^{N} 2^{-2p} b_p \sum_{r=1}^{2^{N-p}} (S_{p,r})^2 \tag{3}$$

where $b_1 \ldots b_N$ are non-negative coefficients. The statistical properties of the model are completely defined given the above and a temperature $T = (k\beta)^{-1}$."

Dyson is able to show that much useful information about the Ising model with long range interactions can be obtained by analyzing the hierarchical model and connecting the hierarchical model to the Ising model through the inequalities of Griffiths [2]. Dyson was not able, nor are we able, to solve the hierarchical model directly. In this paper we