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Quantum Field Theory and the Coloring Problem of Graphs

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Abstract. The ϕ^k theory is compared with the multilinear theory of scalar fields $\phi_1, \phi_2, \ldots, \phi_k$ having the same mass as that of ϕ. In particular, it is shown that Feynman integrals encountered in the ϕ^3 theory are not necessarily present also in the $\phi_1\phi_2\phi_3$ theory, but they are if they correspond to planar Feynman graphs having no tadpole part. Furthermore, a necessary and sufficient condition for the presence of a ϕ^3 Feynman integral in the $\phi_1\phi_2^2$ theory is found. Those considerations are applications of graph theory, especially of the coloring problem of graphs, to Feynman graphs.

1. Introduction and Results

The main question which we discuss in the present paper is as follows: Let $\phi_1, \phi_2, \ldots, \phi_k$ be k scalar quantized fields having an identical mass, which is also equal to the mass of another scalar field ϕ; then, to what extent the multilinear theory of $\phi_1, \phi_2, \ldots, \phi_k$ (its interaction Lagrangian density is $\mathscr{L}_I^{(k)} = \lambda\,\phi_1\phi_2\ldots\phi_k$) is different from the ϕ^k theory ($\mathscr{L}_I = (\lambda/k!):\phi^k:$)? It seems to be generally believed indistinctly that every Feynman integral of the ϕ^k theory would be present also in the $\phi_1\phi_2\ldots\phi_k$ theory, that is, every Feynman graph appearing in the ϕ^k theory[1] could be renamed as a Feynman graph of the $\phi_1\phi_2\ldots\phi_k$ theory. The existence of counterexamples to this conjecture was pointed out explicitly by the present author [1] (see Fig. 1) in connection with the proof of divergence

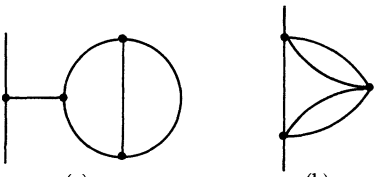

Fig. 1. Examples of ϕ^k Feynman graphs which are not realizable in the $\phi_1\phi_2\ldots\phi_k$ theory [(a) for $k=3$ and (b) for $k=4$; for $k \geqq 5$, add $k-4$ internal lines and $k-4$ external lines to (b)]

[1] Hereafter, we abbreviate a Feynman graph appearing in the ϕ^k theory as a ϕ^k Feynman graph.