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Mixing Properties, Differentiability of the Free Energy and the Central Limit Theorem for a Pure Phase in the Ising Model at Low Temperature

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Abstract. For the Ising model with nearest neighbour interaction it is shown that the spin correlations $\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle$ decrease exponentially as $d(A, B)\to\infty$ in a pure phase when the temperature is well below T_c. This is used to prove that the free energy $F(\beta, h)$ is infinitely differentiable in β and has one sided derivatives in h of all orders for $h=0$. The bounds are also used to prove that the central limit theorem holds for several variables such as e.g. the total energy and the total magnetization of the system, the limit distribution being gaussian with variances determined by the second derivatives of $F(\beta, h)$.

Introduction

We consider the Ising model with nearest neighbour interaction in a finite box Λ on a ν-dimensional square lattice Z^ν. The spin at each point $p\in\Lambda$ takes the values $\sigma_p=\pm 1$, and the energy of a spin configuration is given by

$$-E_\Lambda(\sigma)=\tfrac{1}{2}\sum_{p,q\in\Lambda} J_{p,q}\sigma_p\sigma_q+\sum_{p\in\Lambda}\sigma_p\sum_{q\notin\Lambda}J_{p,q}+H\sum_{p\in\Lambda}\sigma_p \tag{1}$$

where $J_{p,q}=J>0$ if p and q are neighbours and $J_{p,q}=0$ otherwise, and H is the external magnetic field. We are only going to consider the situation where Λ is completely surrounded by $+$ spins, which give rise to the boundary term in the energy. The Boltzmannfactor is $e^{-\frac{E_\Lambda(\sigma)}{kT}}$. We put $\frac{2J}{kT}=\beta$ and $\frac{H}{kT}=h$ and denote the spin correlations $\left\langle\prod_{p\in A}\sigma_p\right\rangle$ by $\langle\sigma_A\rangle_{h,\Lambda}$ $A\subseteqq\Lambda$. The free energy (multiplied by $-kT$) is given by

$$F(\beta,h,\Lambda)=|\Lambda|^{-1}\log\sum_\sigma e^{-\frac{E_\Lambda(\sigma)}{kT}}. \tag{2}$$

When Λ increases to all of Z^ν (in the sense of van Hove) the $\langle\sigma_A\rangle_{h,\Lambda}$ decrease to limits $\langle\sigma_A\rangle_h$ which determine the state of an infinite system